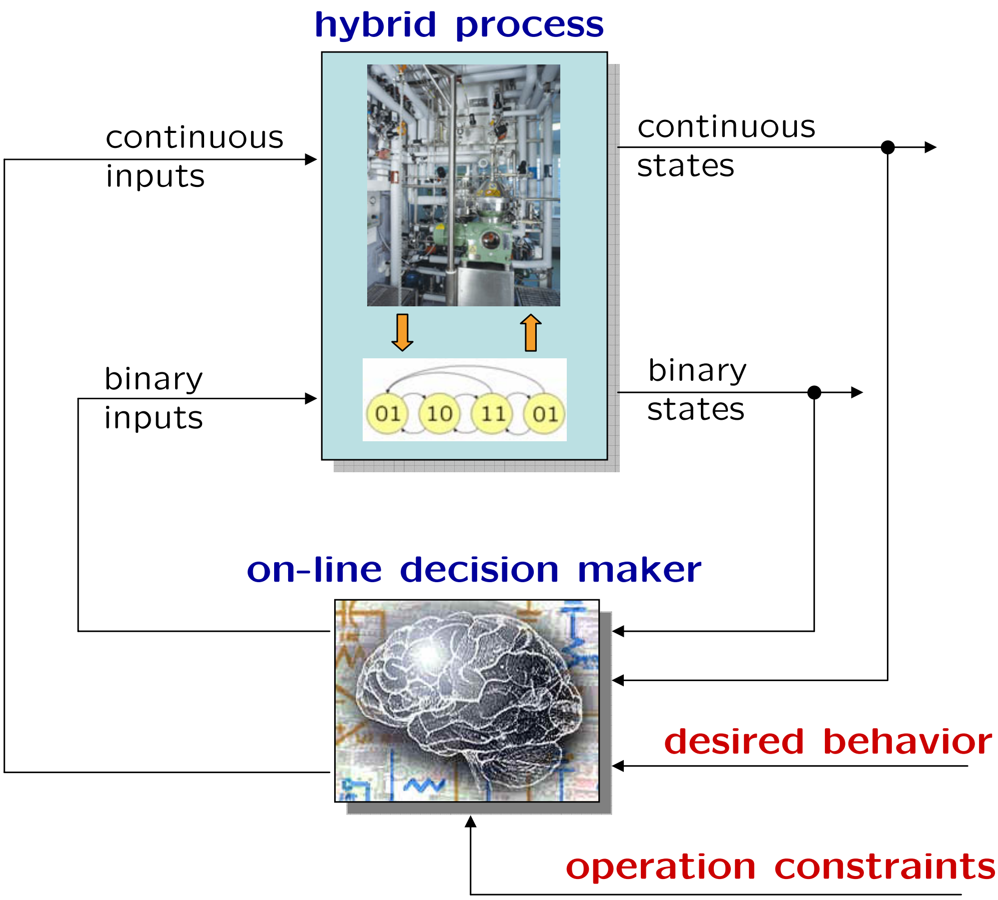
Hybrid Toolbox for Matlab

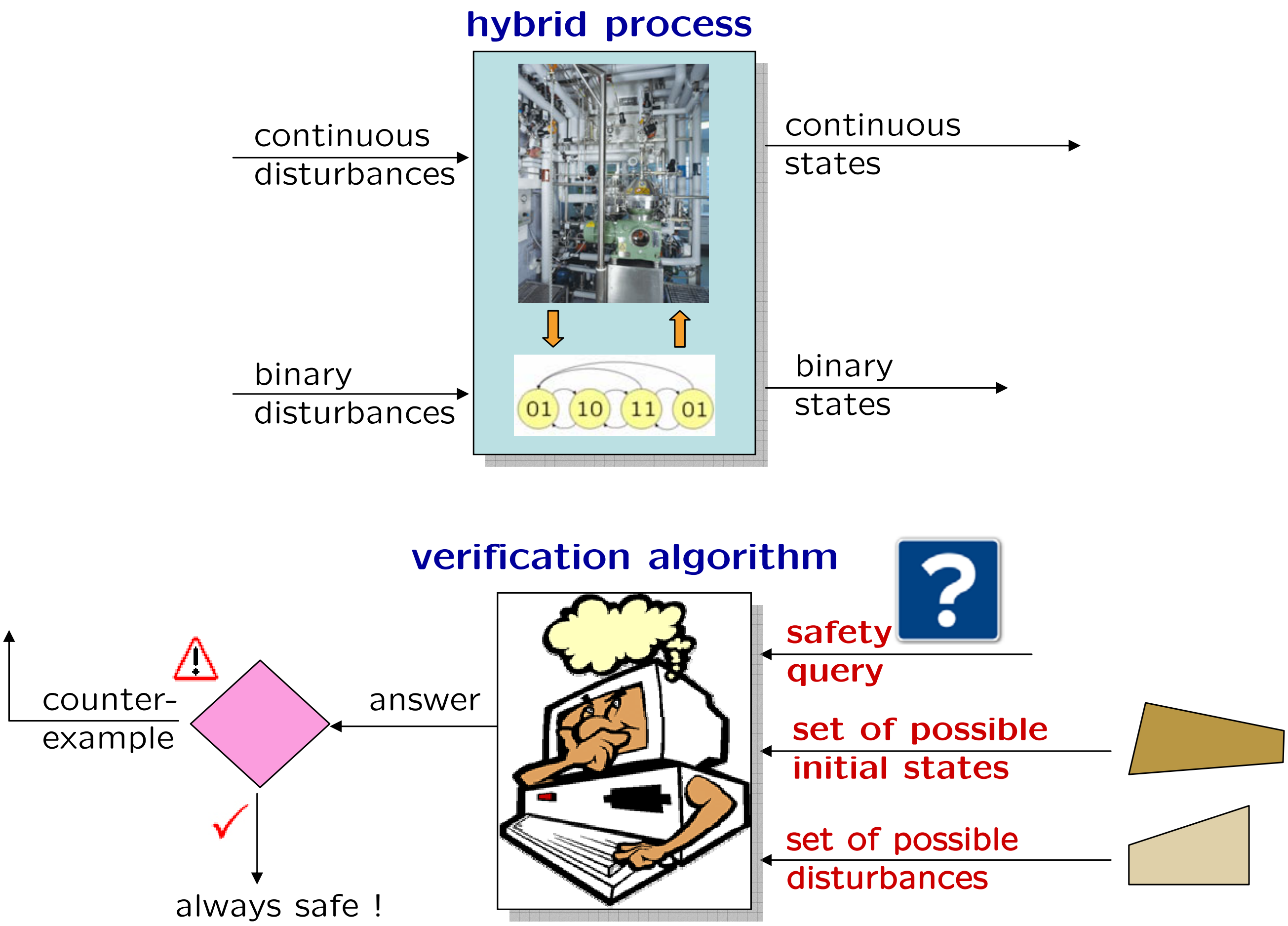
Alberto Bemporad

http://www.dii.unisi.it/hybrid

Hybrid Control Problem

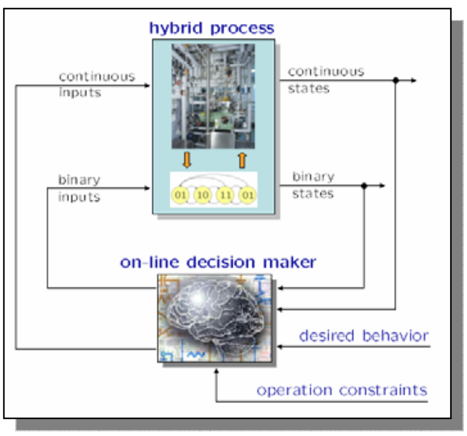


Hybrid Verification Problem

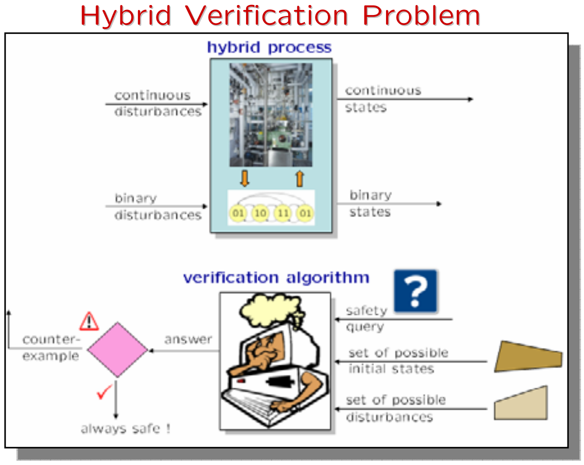


Model-based Optimization Approach

Hybrid Control Problem



Hybrid Verification Problem



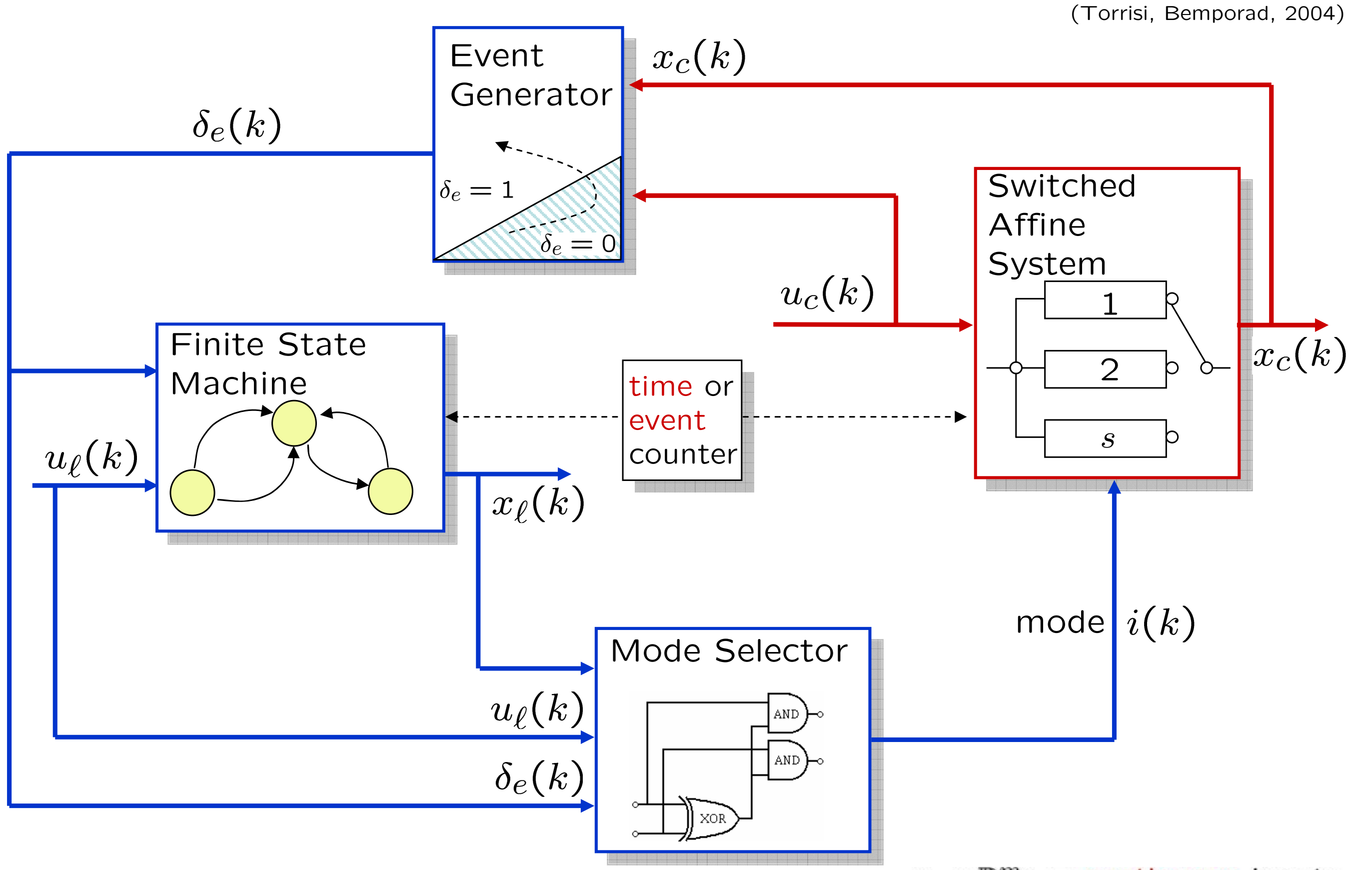
- Need for a <u>hybrid model</u> of the process reproducing the behavior of the process (simulation)

- A model suitable for controller synthesis and verification

- A model for which <u>computational tools</u> can be applied

Hybrid Model: Discrete Hybrid Automaton

(Torrisi, Bemporad, 2004)



$x_c \in \mathbb{R}^{n_r}$ = continuous states

$x_\ell \in \{0,1\}^{n_b}$ = binary states

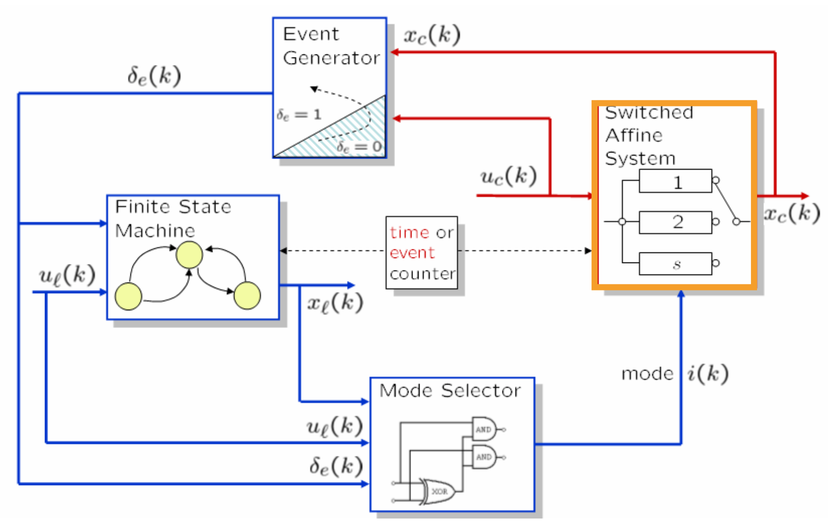
$i(k) \in \{1,\ldots,s\}$ = current mode

$u_c \in \mathbb{R}^{m_r}$ = continuous inputs

$u_\ell \in \{0,1\}^{m_b}$ = binary inputs

$\delta_e \in \{0,1\}^{n_e}$ = event conditions

Switched Affine System

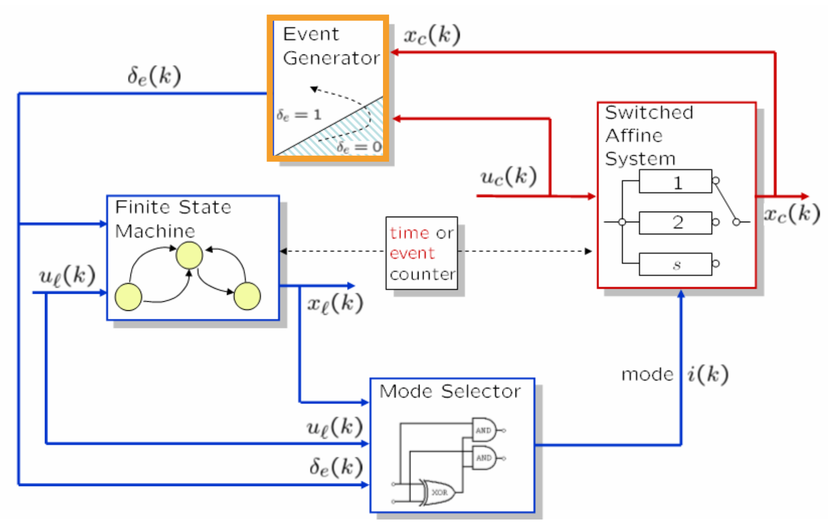


The affine dynamics depend on the current mode $i(k)$:

$$\boxed{x_c(k+1) = A_{i(k)} x_c(k) + B_{i(k)} u_c(k) + f_{i(k)}}$$

$x_c \in \mathbb{R}^{n_c}, \ u_c \in \mathbb{R}^{m_c}$

Event Generator



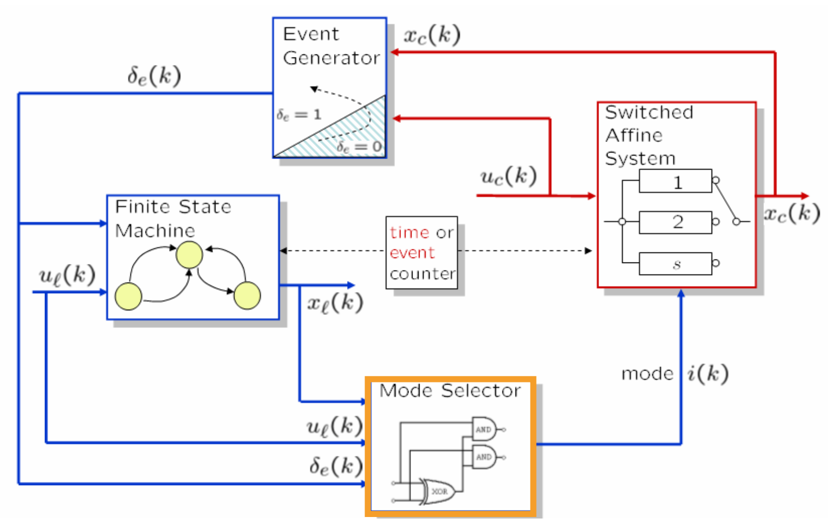
Event variables are generated by linear threshold conditions over continuous states, continuous inputs, and time:

$$\boxed{[\delta_e^i(k) = 1] \leftrightarrow [H^i x_c(k) + K^i u_c(k) \leq W^i]}$$

$x_c \in \mathbb{R}^{n_c},\ u_c \in \mathbb{R}^{m_c},\ \delta_e \in \{0,1\}^{n_e}$

Example: $[\delta=1] \leftrightarrow [x_c(k) \geq 0]$

Mode Selector



The active mode $i(k)$ is selected by a Boolean function of the current binary states, binary inputs, and event variables:

$$\boxed{i(k) = f_{\mathsf{M}}(x_\ell(k), u_\ell(k), \delta_e(k))}$$

$x_\ell \in \{0,1\}^{n_\ell},\ u_\ell \in \{0,1\}^{m_\ell},\ \delta_e \{0,1\}^{n_e}$

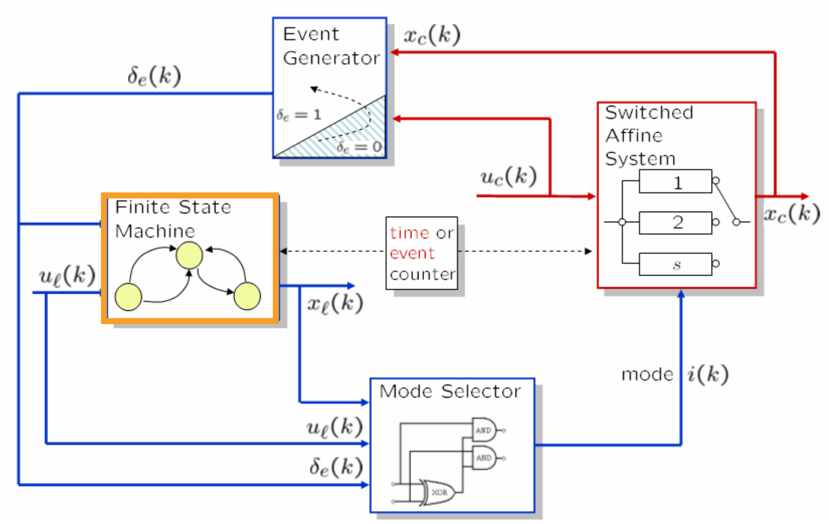
Example:

$$i(k) = \begin{bmatrix} \neg u_\ell(k) \vee x_\ell(k) \\ u_\ell(k) \wedge x_\ell(k) \end{bmatrix} \Longrightarrow$$

u_ℓ / x_ℓ	0	1
0	$i = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$	$i = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$
1	$i = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$	$i = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$

the system has 3 modes

Finite State Machine



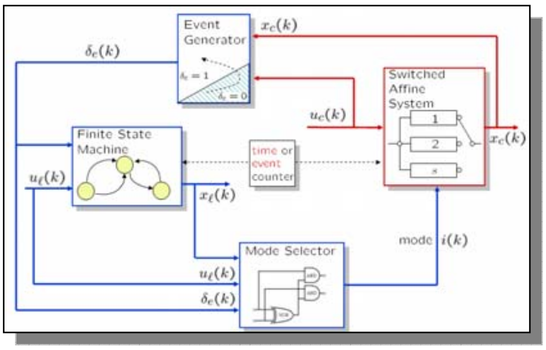
The binary state of the finite state machine evolves according to a Boolean state update function:

$$\boxed{x_\ell(k+1) = f_{\mathrm{B}}(x_\ell(k), u_\ell(k), \delta_e(k))} \quad x_\ell \in \{0,1\}^{n_\ell},\ u_\ell \in \{0,1\}^{m_\ell},\ \delta_e \in \{0,1\}^{n_e}$$

Example: $x_\ell(k+1) = \neg\delta_e(k) \vee (x_\ell(k) \wedge u_\ell(k))$

Computational Hybrid Models

Discrete Hybrid Automaton



HYSDEL

(Torrisi, Bemporad, 2004)

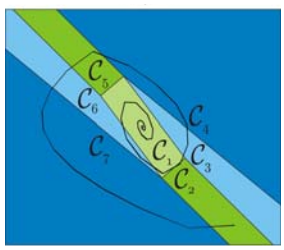
Piecewise Affine (PWA) Systems

$$\begin{array}{rcl} x(k+1) & = & A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) & = & C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ \\ i(k) & \text{s.t.} & H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)} \end{array}$$

MLD2PWA

(Bemporad, 2004)

state+input space



(Sontag 1981)

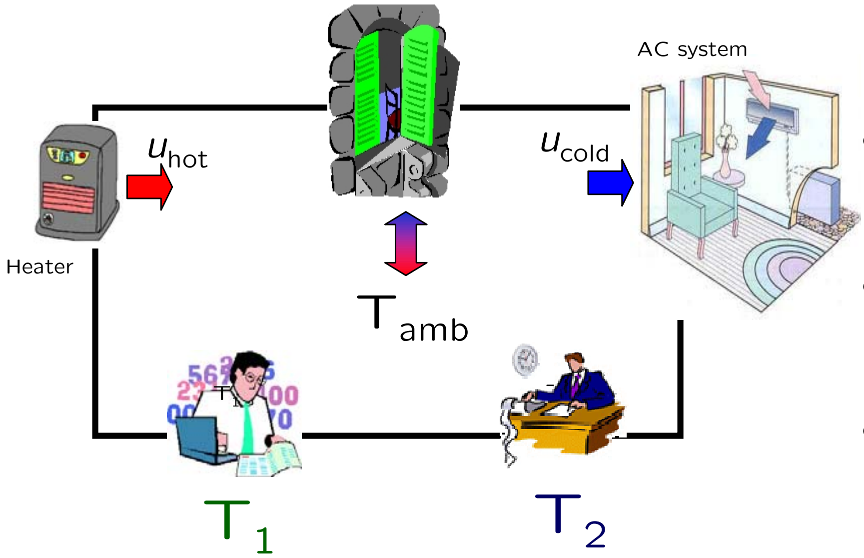
Mixed Logical Dynamical (MLD) Systems

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\ y(t) & = & Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\ E_2\delta(t) + E_3z(t) & \leq & E_4x(t) + E_1u(t) + E_5 \end{array}$$

(Bemporad, Morari 1999)

The translation from DHA to MLD/PWA is done automatically (using symbolic/mathematical programming tools)

Example: Room Temperature



Hybrid Dynamics

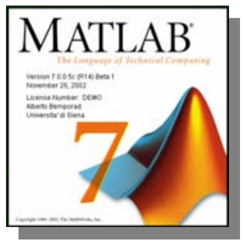
- #1 turns the heater (air conditioning) on whenever he is cold (hot)
- If #2 is cold he turns the heater on, unless #1 is hot
- If #2 is hot he turns the air conditioning on, unless #1 is cold
- Otherwise, heater and air conditioning are off

- $\dot{T}_1 = -\alpha_1(T_1 - T_{\text{amb}}) + k_1(u_{\text{hot}} - u_{\text{cold}})$ (body temperature dynamics of #1)
- $\dot{T}_2 = -\alpha_2(T_2 - T_{\text{amb}}) + k_2(u_{\text{hot}} - u_{\text{cold}})$ (body temperature dynamics of #2)

go to demo **/demos/hybrid/heatcool.m**

HYSDEL Model

```
SYSTEM heatcool {

INTERFACE {
    STATE { REAL T1 [-10,50];
            REAL T2 [-10,50];
          }
    INPUT { REAL Tamb [-10,50];
          }
    PARAMETER {
        REAL Ts, alpha1, alpha2, k1, k2;
        REAL Thot1, Tcold1, Thot2, Tcold2, Uc, Uh;
        }
}
IMPLEMENTATION {
        AUX { REAL uhot, ucold;
              BOOL hot1, hot2, cold1, cold2;
        }
        AD  { hot1  = T1>=Thot1;
              hot2  = T2>=Thot2;
              cold1 = T1<=Tcold1;
              cold2 = T2<=Tcold2;
        }
        DA  { uhot  = {IF cold1 | (cold2 & ~hot1) THEN Uh ELSE 0};
              ucold = {IF hot1 | (hot2 & ~cold1) THEN Uc ELSE 0};
        }
        CONTINUOUS { T1 = T1+Ts*(-alpha1*(T1-Tamb)+k1*(uhot-ucold));
                     T2 = T2+Ts*(-alpha2*(T2-Tamb)+k2*(uhot-ucold));
        }
    }
}
```



Hybrid Toolbox for Matlab

(Bemporad, 2003-2005)

http://www.dii.unisi.it/hybrid/toolbox

```
>>S=mld('heatcoolmodel',Ts)
```

get the MLD model in Matlab

```
>>[XX,TT]=sim(S,x0,U);
```

simulate the MLD model

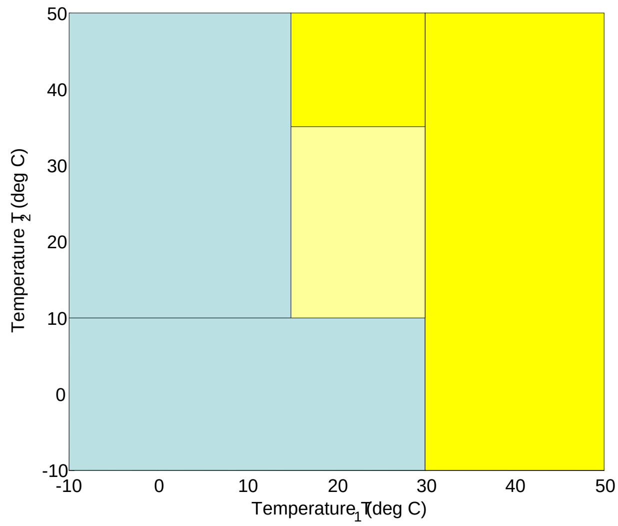
Hybrid PWA Model

- PWA model

$$\begin{aligned} x(k+1) &= A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)} \\ y(k) &= C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)} \\ i(k) \;\; \text{s.t.} & \;\; H_{i(k)}x(k) + J_{i(k)}u(k) \le K_{i(k)} \end{aligned}$$

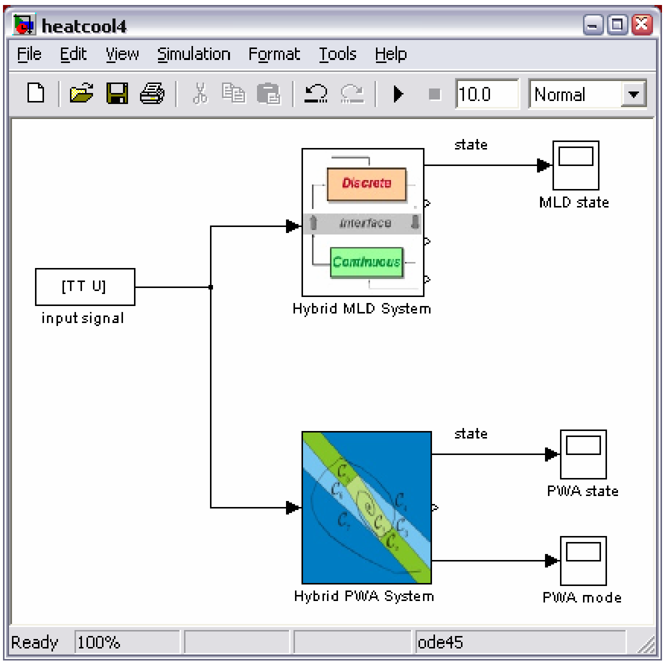
- 2 continuous states:
  (temperatures T_1,T_2)
- 1 continuous input:
  (room temperature T_{amb})
- 5 polyhedral regions
  (partition does not depend on input)

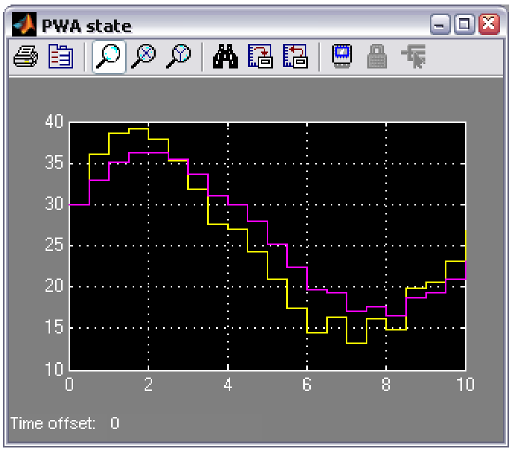
```
>>P=pwa(S);
```

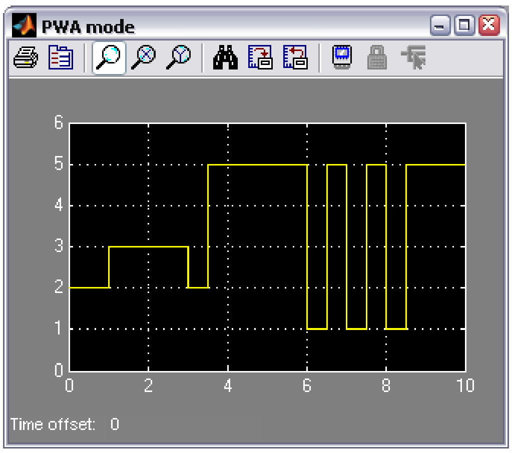


$$\begin{aligned} u_{\text{hot}} &= 0 \\ u_{\text{cold}} &= 0 \end{aligned} \qquad \begin{aligned} u_{\text{hot}} &= 0 \\ u_{\text{cold}} &= \bar{U}_C \end{aligned} \qquad \begin{aligned} u_{\text{hot}} &= \bar{U}_H \\ u_{\text{cold}} &= 0 \end{aligned}$$

Simulation in Simulink







Verification of DHA/MLD/PWA

Verification Algorithm

- **QUERY**: Is the target set X_f reachable after N steps from some initial state $x_0 \in X_0$ for some input profile $u \in U$?

- Computation: Solve the mixed-integer linear program (MILP)

$$
\begin{array}{ll}
\min & 0 \\
\text{s.t.} & \left\{ \begin{array}{rcl}
x(k+1) & = & Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) \\
E_2 \delta(k) & + & E_3 z(k) \leq E_1 u(k) + E_4 x(k) + E_5 \\
S_u u(k) & \leq & T_u \quad (u(k) \in U) \\
 & & k = 0, 1, \ldots, N-1 \\
 & & \\
S_0 x(0) & \leq & T_0 \quad (x(0) \in X_0) \\
S_f x(N) & \leq & T_f \quad (x(N) \in X_f)
\end{array} \right.
\end{array}
$$

with respect to u(0),δ(0),z(0),..., u(N-1),δ(N-1),z(N-1),x(0)

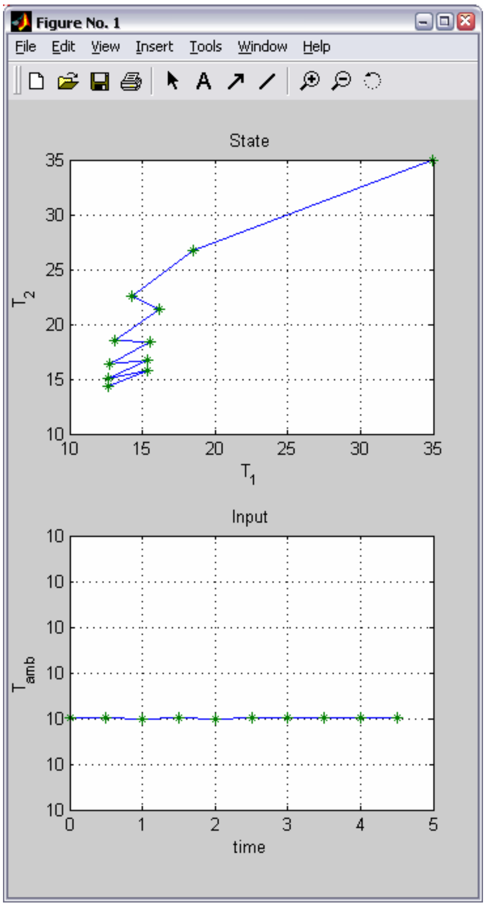
- Alternative solutions:
  - Exploit the special structure of the problem and use polyhedral computation. (Torrisi, 2003)
  - Use abstractions (LPs) + SAT solvers (Giorgetti, Pappas, 2005)

Verification Example

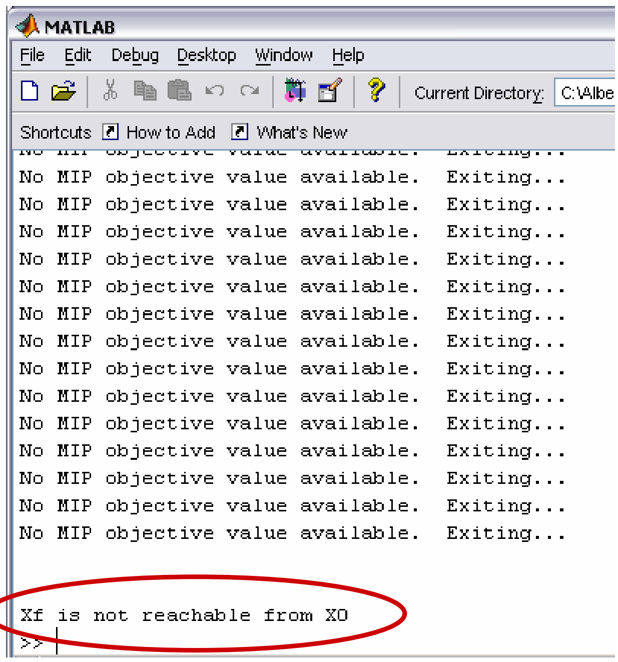
- MLD model: room temperature system
- $X_f = \{\left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right] : 10 \leq T_1, T_2 \leq 15\}$ (set of unsafe states)
- $X_0 = \{\left[\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right] : 35 \leq T_1, T_2 \leq 40\}$ (set of initial states)
- $U = \{T_{\text{amb}} : 10 \leq T_{\text{amb}} \leq 30\}$ (set of possible inputs)
- N=10 (time horizon)

```
>>[flag,x0,U]=reach(S,N,Xf,X0,umin,umax);
>>[flag,x0,U]=reach(S,[1 N],Xf,X0,umin,umax);
```

Verification Example



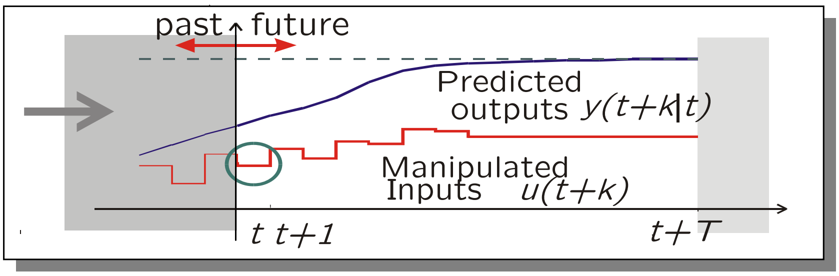
$U = \{T_{\text{amb}} : 10 \leq T_{\text{amb}} \leq 30\}$



$U = \{T_{\text{amb}} : 20 \leq T_{\text{amb}} \leq 30\}$

Controller Synthesis

Control Strategy: MPC



Model
Predictive (MPC)
Control

- At time t solve with respect to $U \triangleq \{u(t), u(t+1), \ldots, u(t+T-1)\}$ the finite-horizon open-loop, optimal control problem:

$$\min_{u(t),\ldots,u(t+T-1)} \sum_{k=0}^{T-1} \|R(y(t+k|t) - r(t+k))\|_p + \|Qu(t+k)\|_p$$

$$\text{subject to} \begin{cases} \text{MLD or PWA model} \\ x(t|t) = x(t) \end{cases}$$

$p = 1, 2, \infty \qquad \|v\|_2 = v'v \quad \|v\|_\infty = \max |v_i| \quad \|v\|_1 = \sum v_i$

- Apply only $u(t) = u^*(t)$ (discard the remaining optimal inputs);
- Repeat the whole optimization at time $t+1$

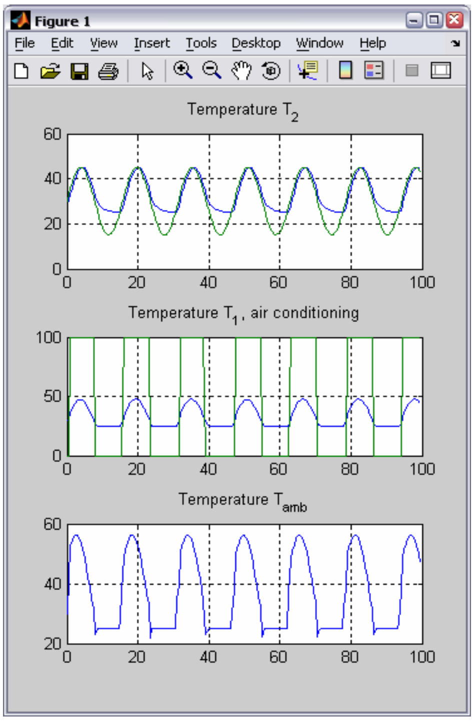
Hybrid MPC - Example

```
>>refs.x=2;    % just weight state #2
>>Q.x=1;
>>Q.rho=Inf;   % hard constraints
>>Q.norm=2;    % quadratic costs
>>N=2;         % optimization horizon
>>limits.xmin=[25;-Inf];
```

$$\min \sum_{k=1}^{2} (x_2(k) - r)^2$$
$$\text{s.t.} \quad x_1(k) \geq 25 \quad k = 1, 2$$
MLD model

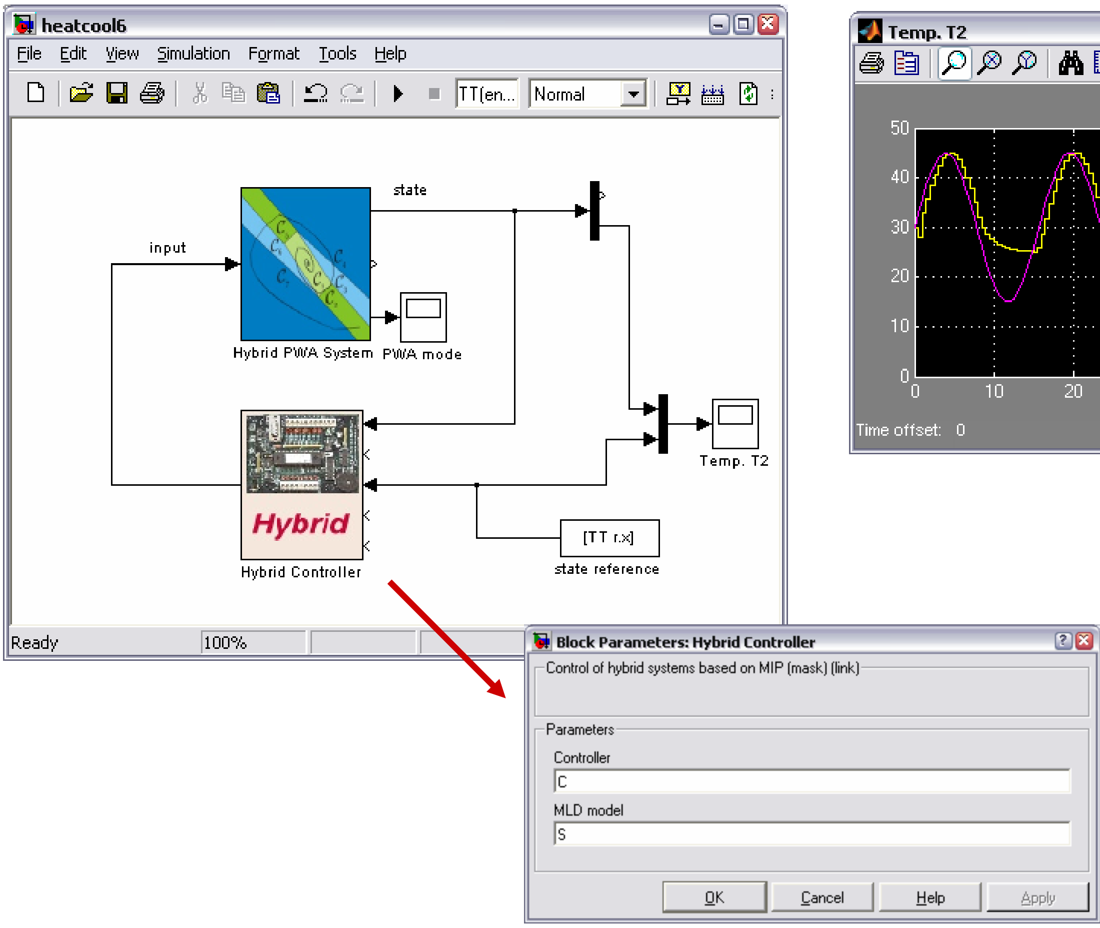
```
>>C=hybcon(S,Q,N,limits,refs);
```

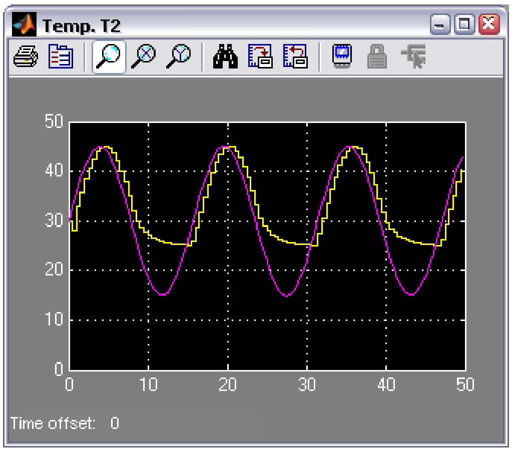
```
>> C

Hybrid controller based on MLD model S <heatcoolmodel.hys>

  2 state measurement(s)
  0 output reference(s)
  0 input reference(s)
  1 state reference(s)
  0 reference(s) on auxiliary continuous z-variables

 20 optimization variable(s) (8 continuous, 12 binary)
 46 mixed-integer linear inequalities
sampling time = 0.5, MILP solver = 'glpk'

Type "struct(C)" for more details.
>>
```




```
>>[XX,UU,DD,ZZ,TT]=sim(C,S,r,x0,Tstop);
```

Hybrid MPC - Example





On-Line vs Off-Line Optimization

$$\min_U J(U, x(t)) = \sum_{k=0}^{T-1} \|Rx(t+k|t)\|_p + \|Qu(t+k)\|_p$$
$$\text{subject to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

- On-line optimization: given $x(t)$ solve the problem at each time step t.

Mixed-Integer Linear/Quadratic Program (MILP/MIQP)

  - ➢ Good for large sampling times (e.g., 1 h) / expensive hardware ...

  ... but not for fast sampling (e.g. 10 ms) / cheap hardware !

- Off-line optimization: solve the MILP/MIQP <u>for all</u> $x(t)$

$$\min_\zeta J(\zeta, x(t)) = \begin{cases} f'\zeta & \infty/1\text{-norms} \\ \zeta' H \zeta + f'\zeta & \text{quadratic forms} \end{cases}$$
$$\text{s.t. } G\zeta \leq W + Fx(t)$$

multi-parametric programming

Explicit Hybrid MPC

$$\min_U J(U, x, r) = \sum_{k=0}^{T-1} \|R(y(k) - r)\|_p + \|Qu(k)\|_p$$
$$\text{subject to} \begin{cases} \text{PWA model} \\ x(0) = x \end{cases}$$

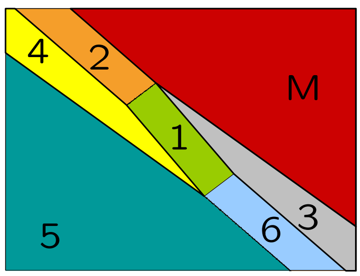
- Solution $u(x,r)$ found via a combination of
  - Dynamic programming or enumeration of feasible mode sequences, multiparametric linear or quadratic programming, and polyhedral computation.

(Borrelli, Baotic, Bemporad, Morari, 2003)
(Mayne, ECC 2001)
(Alessio, Bemporad, 2005)

- **The MPC controller is piecewise affine in x,r**

$$u(x,r) = \begin{cases} F_1 x + E_1 r + g_1 & \text{if } H_1 \left[\begin{smallmatrix} x \\ r \end{smallmatrix}\right] \leq K_1 \\ \vdots & \vdots \\ F_M x + E_M r + g_M & \text{if } H_M \left[\begin{smallmatrix} x \\ r \end{smallmatrix}\right] \leq K_M \end{cases}$$

(x,r)-space



<u>Note</u>: in the quadratic case the partition may not be fully polyhedral

Explicit Hybrid MPC - Example

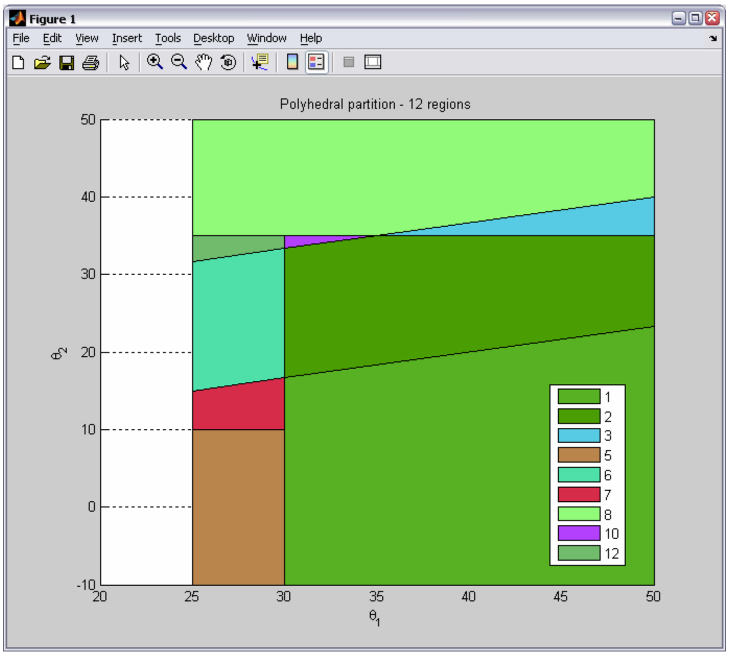
```
>>E=expcon(C,range,options);
```

```
>> E

Explicit controller (based on hybrid controller C)
  3 parameter(s)
  1 input(s)
 12 partition(s)
sampling time = 0.5

The controller is for hybrid systems (tracking)
This is a state-feedback controller.

Type "struct(E)" for more details.
>>
```

$$\min \sum_{k=1}^{2} x_2^2(k)$$

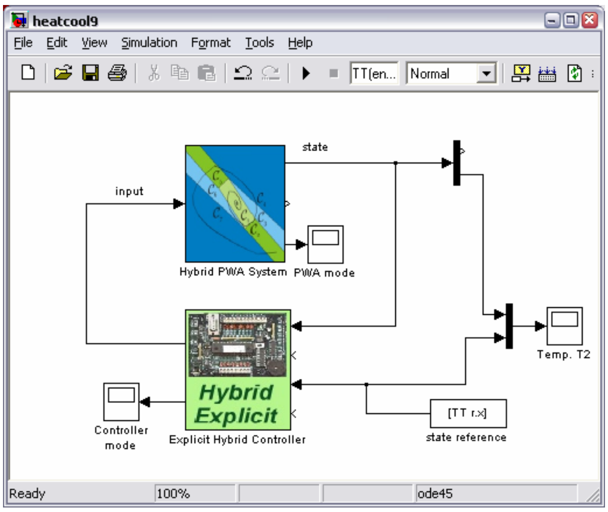
$$\text{s.t. } x_1(k) \geq 25 \quad k = 1,2$$

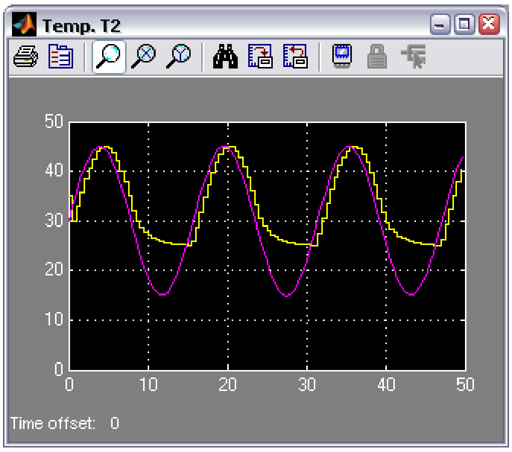
PWA model

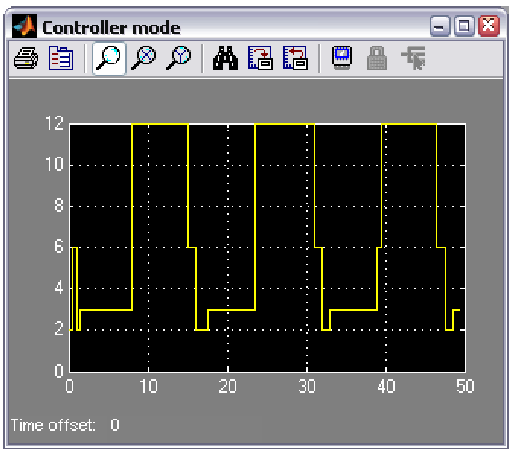


Section in the (T_1, T_2)-space for $T_{ref} = 30$

Explicit Hybrid MPC - Example





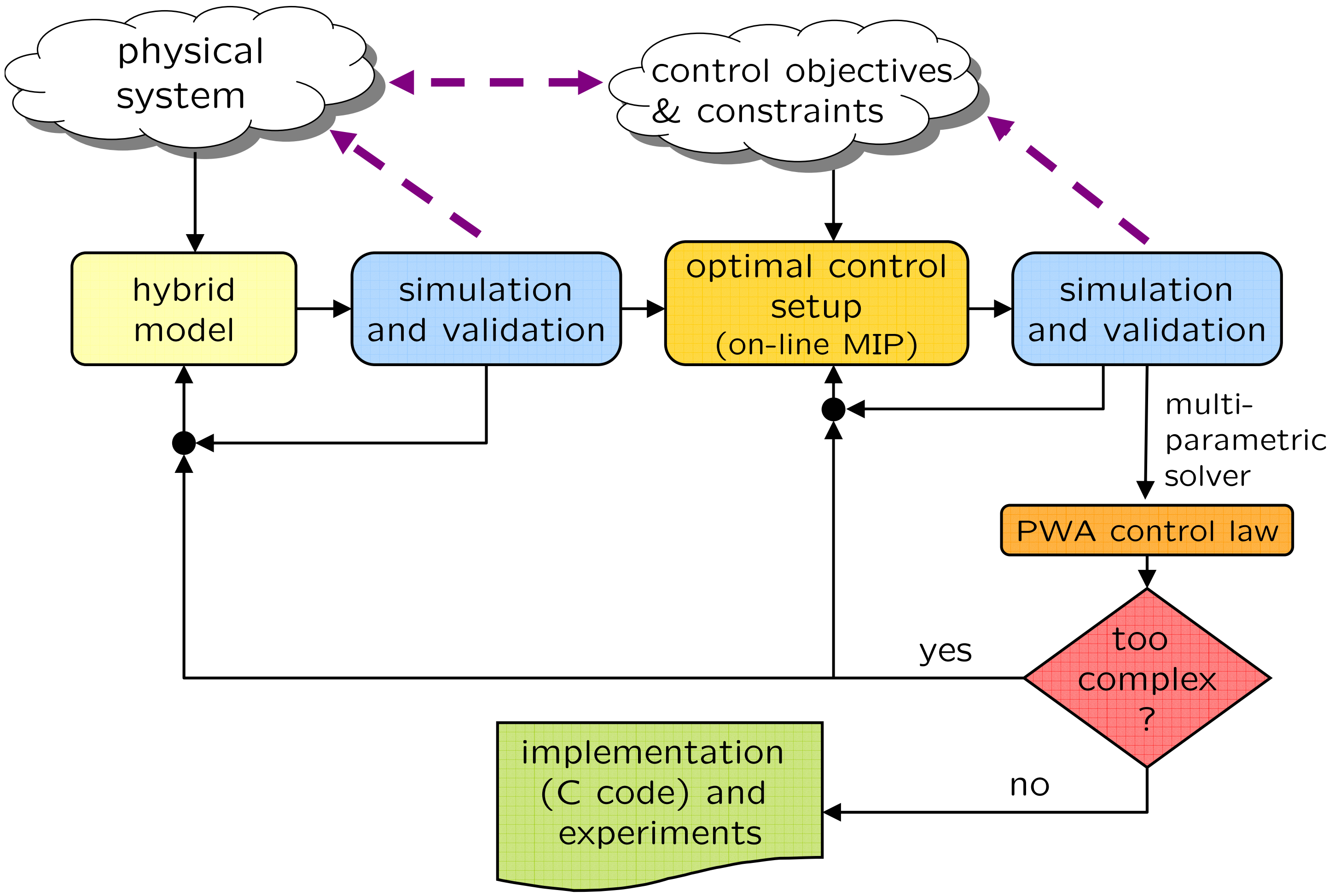


Generated C-code

utils/expcon.h

```
#define EXPCON_REG 12
#define EXPCON_NTH 3
#define EXPCON_NYM 2
#define EXPCON_NH 72
#define EXPCON_NF 12
static double EXPCON_F[]={
    -1,0,0,0,-1,0,
    -1,-1,-1,-1,-1,-1,0,-3,-3,
    -3,0,-3,0,0,0,0,0,
    0,0,4,4,4,0,4,0,0,
    0,0,0,0};

static double EXPCON_G[]={
    101.6,1.6,1.6,-1.6,98.4001,0,100,51.6,
    101.6,51.6,48.4,50};

static double EXPCON_H[]={
    0,0,0,-0.00999999,0,-0.0333333,
    0.02,0.00999999,-0.02,0,0,-0.0333333,0.02,0.00999999,
    0,0,-0.02,0.02,0,-1,0.00999999,0,
```

Hybrid Control Design Flow

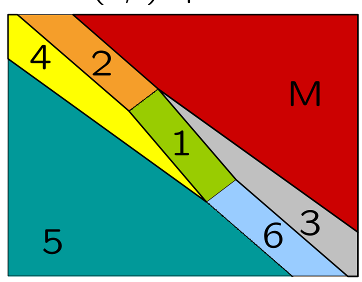


Conclusions

- **Discrete hybrid automata** are simple yet versatile models of hybrid systems, and lead immediately computationally-useful models
- **Optimization-based control** handles performance specs and constraints in a natural and direct way. Quite complex hybrid systems can be controlled using on-line mixed-integer programming
- **Piecewise affine MPC controllers** can be synthesized, off-line, and implemented as look-up tables of linear gains

$$u(x,r) = \begin{cases} F_1 x + E_1 r + g_1 & \text{if} \quad H_1 \left[\begin{smallmatrix} x \\ r \end{smallmatrix}\right] \leq K_1 \\ \vdots & \quad \vdots \\ F_M x + E_M r + g_M & \text{if} \quad H_M \left[\begin{smallmatrix} x \\ r \end{smallmatrix}\right] \leq K_M \end{cases}$$

(x,r)-space



- **Hybrid Toolbox for Matlab** available to assist controller design: modeling, simulation, verification, MPC, code generation

The End

http://www.dii.unisi.it/hybrid/toolbox