Stochastic hybrid systems

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Hybrid dynamics

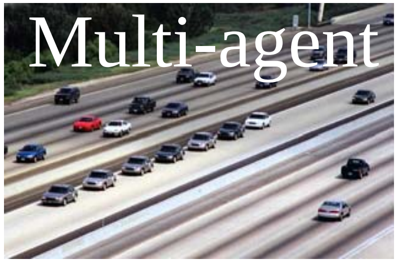
Discrete and continuous interactions

Flight plan
FMS modes

Network topology
Quantization

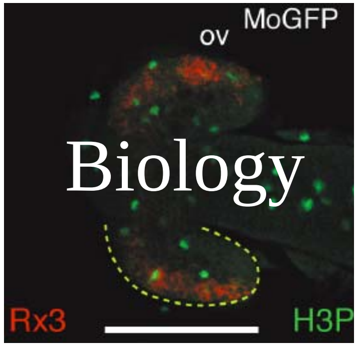
Coordination
communication

Gene activation/
inhibition



Aircraft
motion

Network delays
Controlled state



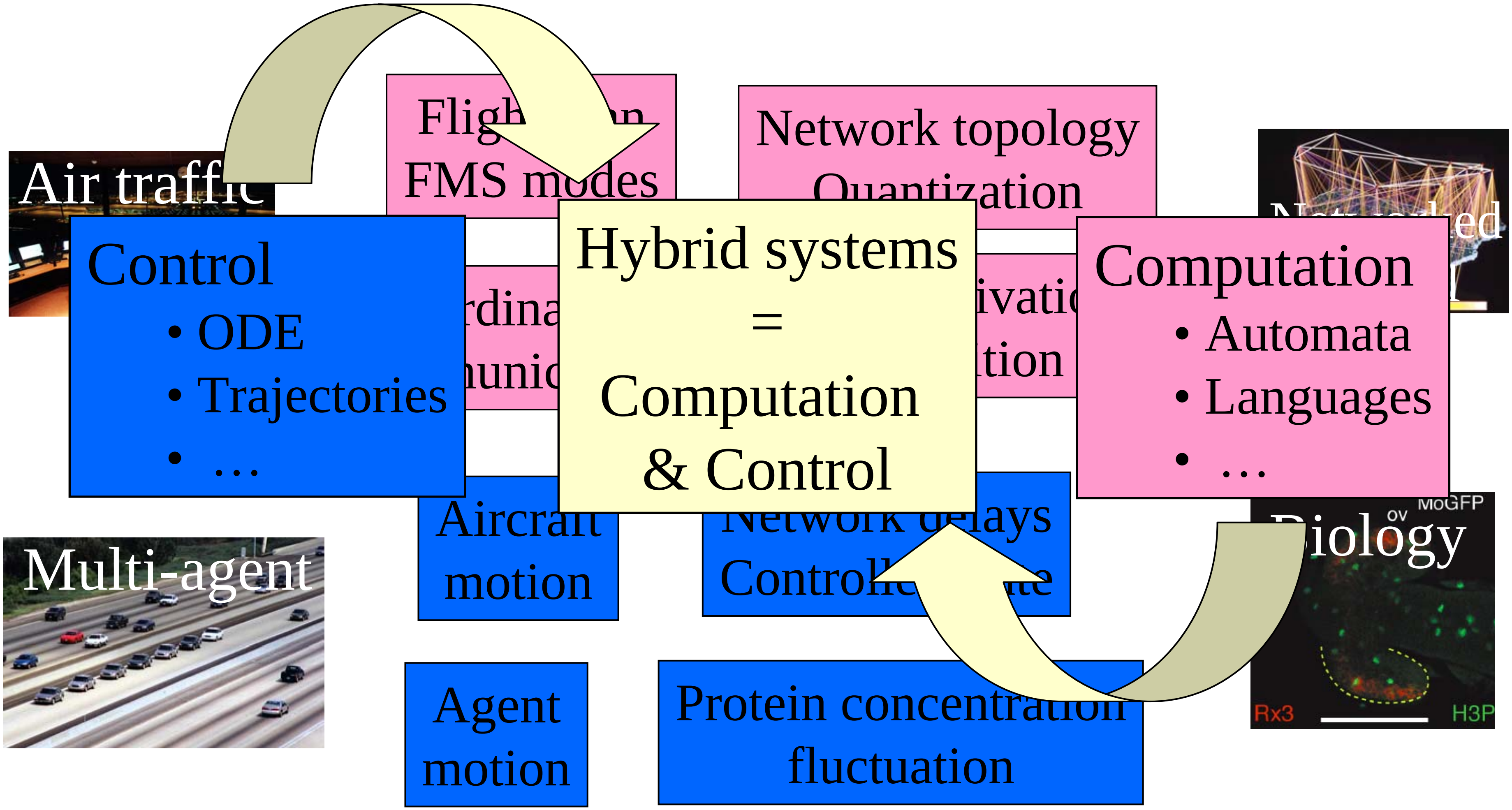
Agent
motion

Protein concentration
fluctuation

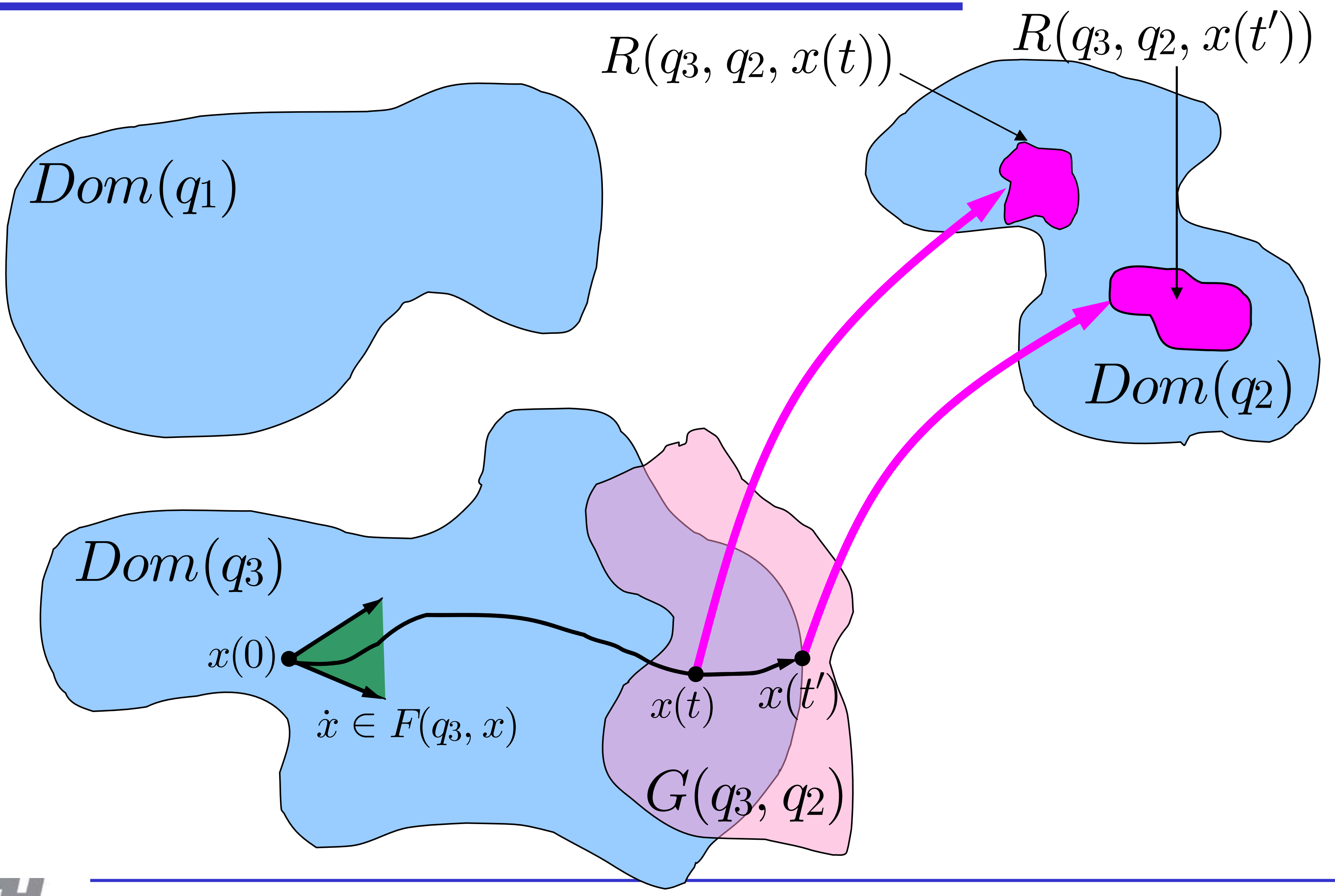
Hybrid dynamics

- Both continuous and discrete state and input
- Interleaving of discrete and continuous
  - Evolve continuously
  - Then take a jump
  - Then evolve continuously again
  - Etc.
- Tight coupling
  - Discrete evolution depends on continuous state
  - Continuous evolution depends on discrete state
- Required new paradigm

Hybrid systems



Solutions



$R(q_3, q_2, x(t))$
$R(q_3, q_2, x(t'))$
$Dom(q_1)$
$Dom(q_2)$
$Dom(q_3)$
$x(0)$
$\dot{x} \in F(q_3, x)$
$x(t)$
$x(t')$
$G(q_3, q_2)$

Uncertainty

- Solutions "declarative" (cf. "imperative")
- Allows uncertainty
  - Select any $(q, x) \in Init$
  - Multiple continuous flow directions $\dot{x} \in F(q, x)$
  - Multiple discrete state destinations

$$G(q, q') \cap G(q, q'') \neq \emptyset$$

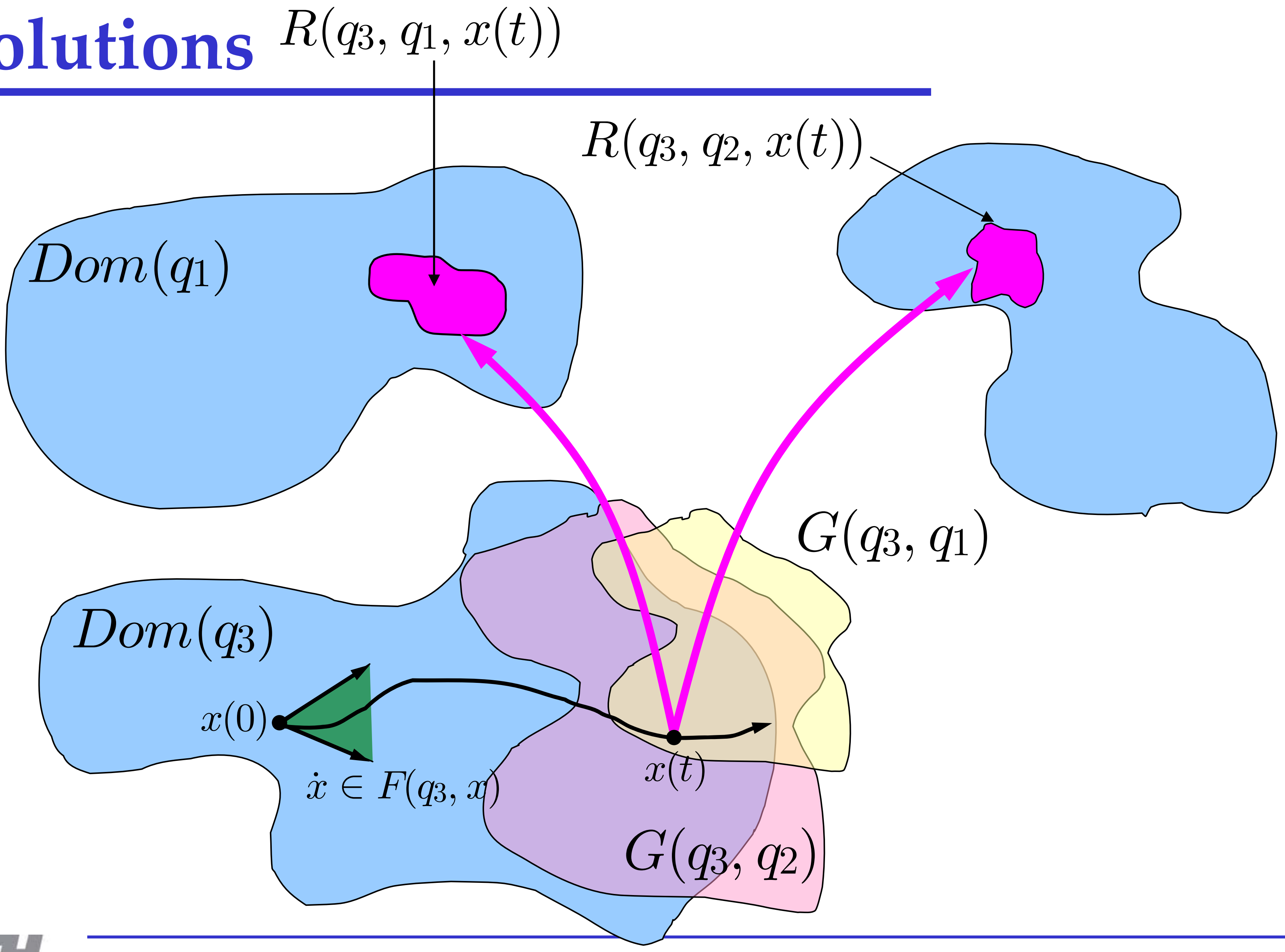
  - Multiple continuous state destinations

$$x' \in R(q, q', x)$$

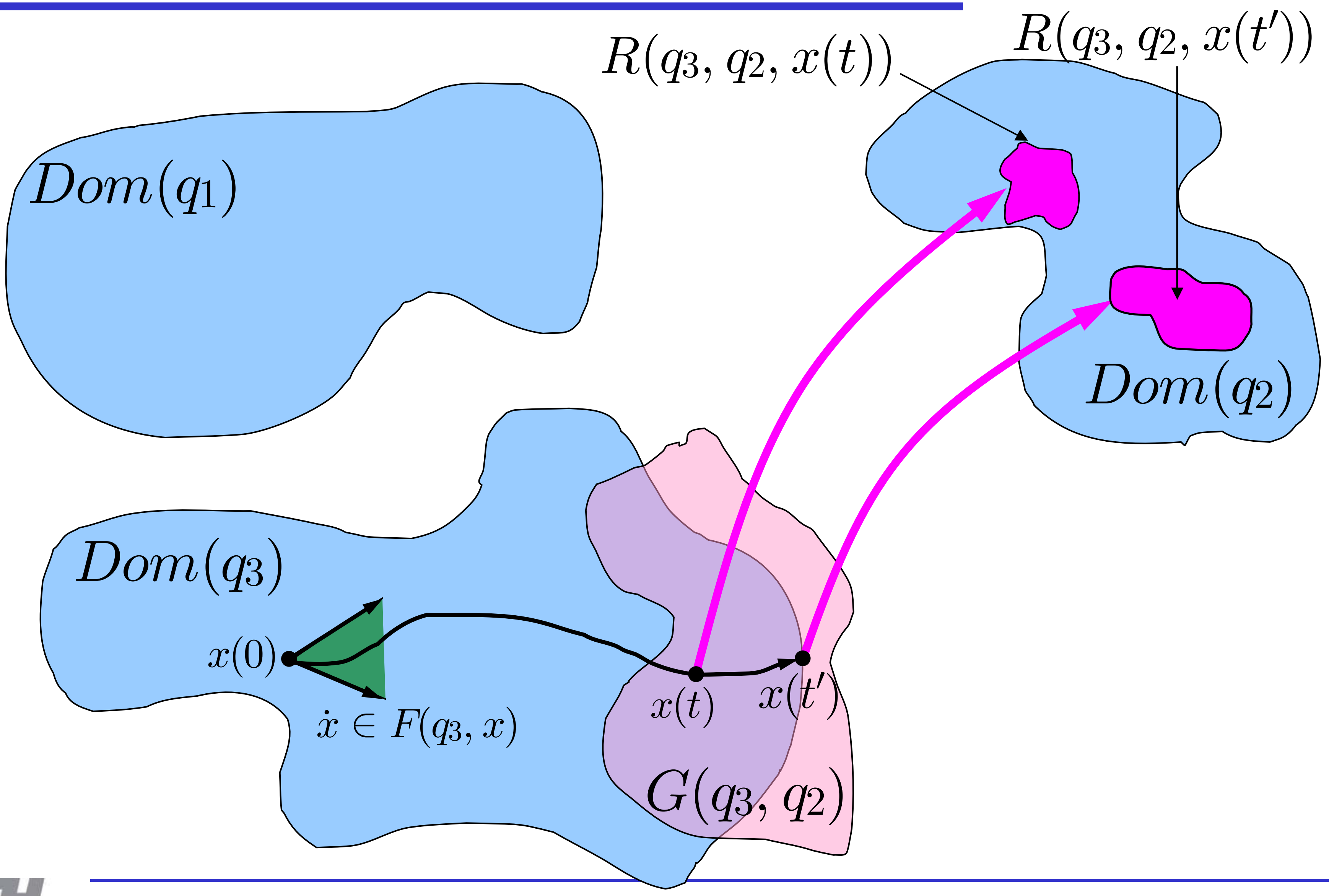
  - Choice between flowing and jumping

$$x \in Dom(q) \cap G(q, q')$$

Solutions



Solutions



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$G(q_3, q_2)$

Non determinism

- Choice typically non-deterministic
  - Many solutions possible
  - Nothing to distinguish one from the others
  - E.g. which is likely and which is not
- Only yes/no questions can be answered
- Dealt with in "worst case" framework, e.g.
  - Reachability → Pursuit evasion games
  - Optimality → Robust control
  - Stability → Practical stability/stabilization
- OK for many applications
- Is it enough in general?

Example: Air traffic safety

Is a fatal accident possible in the current air traffic system? → YES!

Is this an interesting question? → NO!

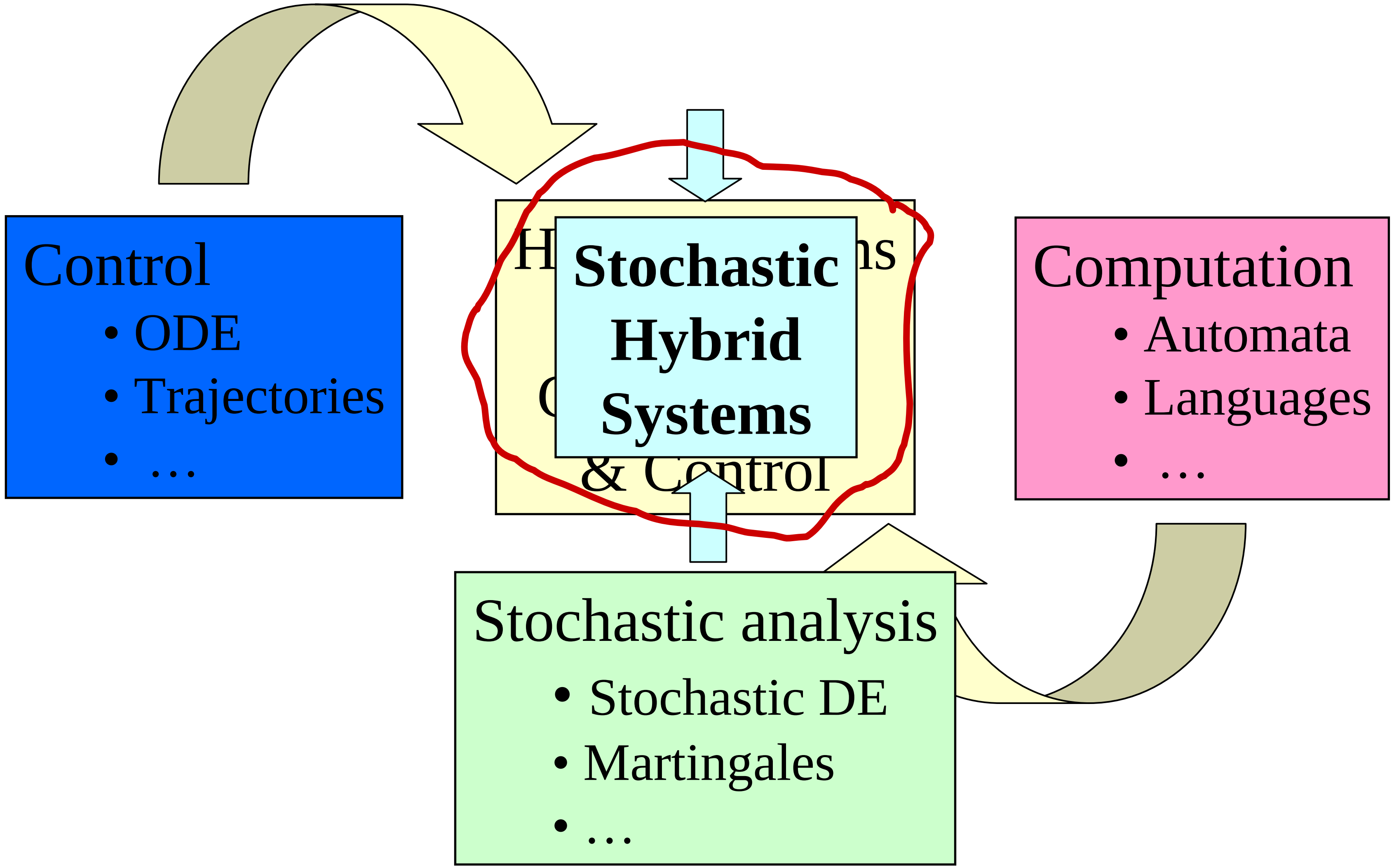
What it is the probability of a fatal accident?

How can this probability be reduced?

→ Much more difficult!

Stochastic hybrid systems

- Answering (or even asking) these questions requires (yet another) paradigm shift
- Richer models to allow probabilities
  - Continuous evolution
  - Timing of discrete transitions
  - Destination of discrete transitions
- Stochastic hybrid systems
  - Hybrid systems + probability



Control
• ODE
• Trajectories
• …
Stochastic Hybrid Systems
& Control
Computation
• Automata
• Languages
• …
Stochastic analysis
• Stochastic DE
• Martingales
• …

Outline

1. Modeling
    1. Classes of models
    2. Comparison
2. Analysis and control
    1. Overview of analysis and control problems
    2. Reachability
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Disclaimer

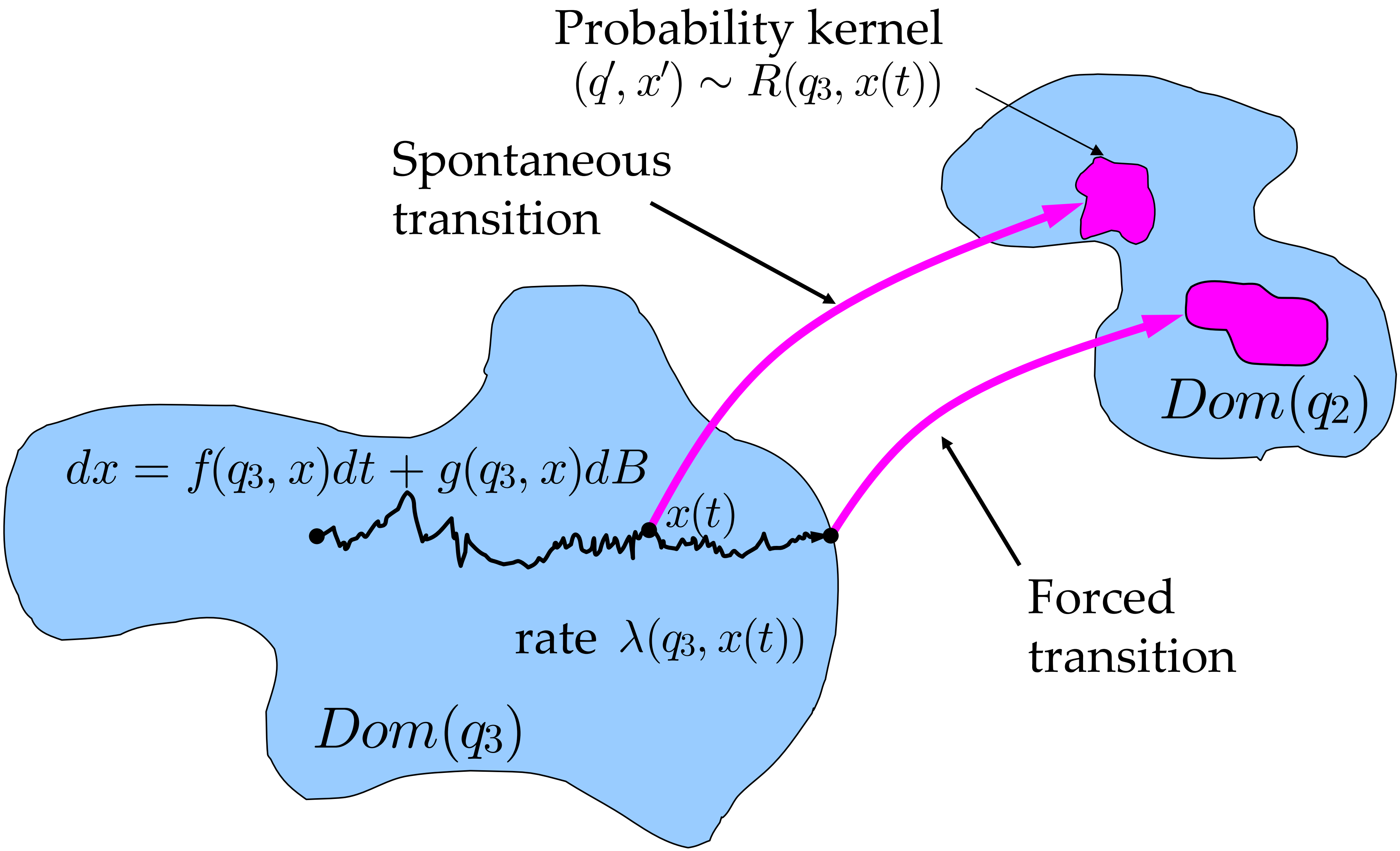
- Very technical field
- In some areas, general understanding just emerging
- Omit technical details, stick to big picture
- Indicate where knowledge ends

- Discussion in continuous time
- In discrete time issues are similar, technical problems are usually simpler
- References included at the end

Classes of SHS

- Autonomous or with inputs
  - Modeling and analysis, stability, reachability
  - Composition, optimal control, stabilization, etc.
- Stochastic uncertainty
  - Initial condition: probability distribution
  - Continuous evolution (e.g. SDE)
  - Destination of discrete transitions: probability distribution, depends on state before transition
  - Choice between jumping and flowing
    - Markovian jumps with rates that depend on continuous state (spontaneous transitions)
    - Boundary hits (forced transitions)

Solutions



Probability kernel
$(q', x') \sim R(q_3, x(t))$
Spontaneous transition
$dx = f(q_3, x)dt + g(q_3, x)dB$
$x(t)$
rate $\lambda(q_3, x(t))$
$Dom(q_3)$
$Dom(q_2)$
Forced transition

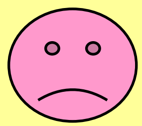
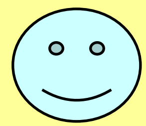
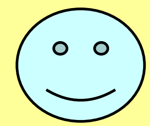
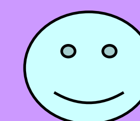
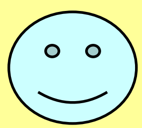
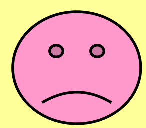
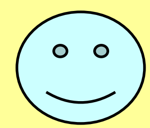
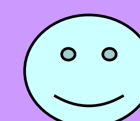
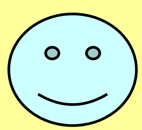
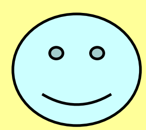
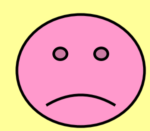
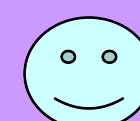
Classes of SHS

- Ingredients present in different combinations
- More general models preferable
  - Properties inherited by special classes
- Fewer properties known for general models
  - Existence, CADLAG
  - Markov and strong Markov properties
  - Generator of the process
  - Characterization of value functions
  - Invariant distributions, stability
- Important in theory and in practice
  - E.g. theoretical foundations of Monte-Carlo simulation

Modeling frameworks

- To illustrate the issues consider three classes of stochastic hybrid modeling frameworks
  - Piecewise Deterministic Processes (PDP) (Davis 1980's)
  - Switching diffusion processes (SDP) (Ghosh et.al. early 1990's)
  - "Stochastic hybrid systems" (SHS) (Hu et.al. late 1990's)
- Restrict attention to autonomous versions
- Controlled versions of PDP and SDP exist

Classification of some SHS [15]

	PDP [3,4]	SDP [5,6]	SHS [8]	GSHP [2,14]
Stochastic continuous evolution				
Forced transitions & continuous reset				
Spontaneous transitions				

Known properties

- PDP:
  - Well posedness, strong Markov property
  - Generator, domain characterization
  - Optimal control, stability
- SDP
  - Well posedness, strong Markov property
  - Generator, domain characterization
  - Optimal control, stability
- SHS
  - Markov property
- GSHP
  - Well posedness, strong Markov property
  - Generator (sort of)

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Analysis and design questions

Modeling

Analysis | **Design**

Analysis	Design
Stability	Stabilization
Language verification	Language "synthesis"
	Optimal control
…	…

Safety = Reachability

Stability and stabilization

- Different notions of stochastic stability
  - Existence of invariant measures
  - Moment asymptotic stability
  - Almost sure asymptotic stability
  - …
- Sufficient conditions based on Lyapunov functions
- Well studied for classes of SDP
- Studied in the 1980's for PDP
- Very little known about other SHS classes

Optimal control

- Introduce control variables to
  - Drive continuous motion
  - Influence discrete transition rate
  - Force discrete transitions
  - Influence discrete transition destination
- Different combos in different approaches
- Introduce admissible control policies
  - Feedback
  - Markov
  - Non-anticipative
- Introduce cost function to assign cost to control policy

Optimal control

- Usually expected value

$$\mathbb{E}\{\int_0^\infty l(x(t), u(t))dt + \sum_{i=0}^\infty c(x(\tau_i), u(\tau_i))\}$$

- Minimize over all admissible control policies
- Define value function
- Develop dynamic programming principle
- Characterize value function as PDE solution
  - Coupled second order for SDP
  - First order with boundary conditions for PDP
  - ??? for others

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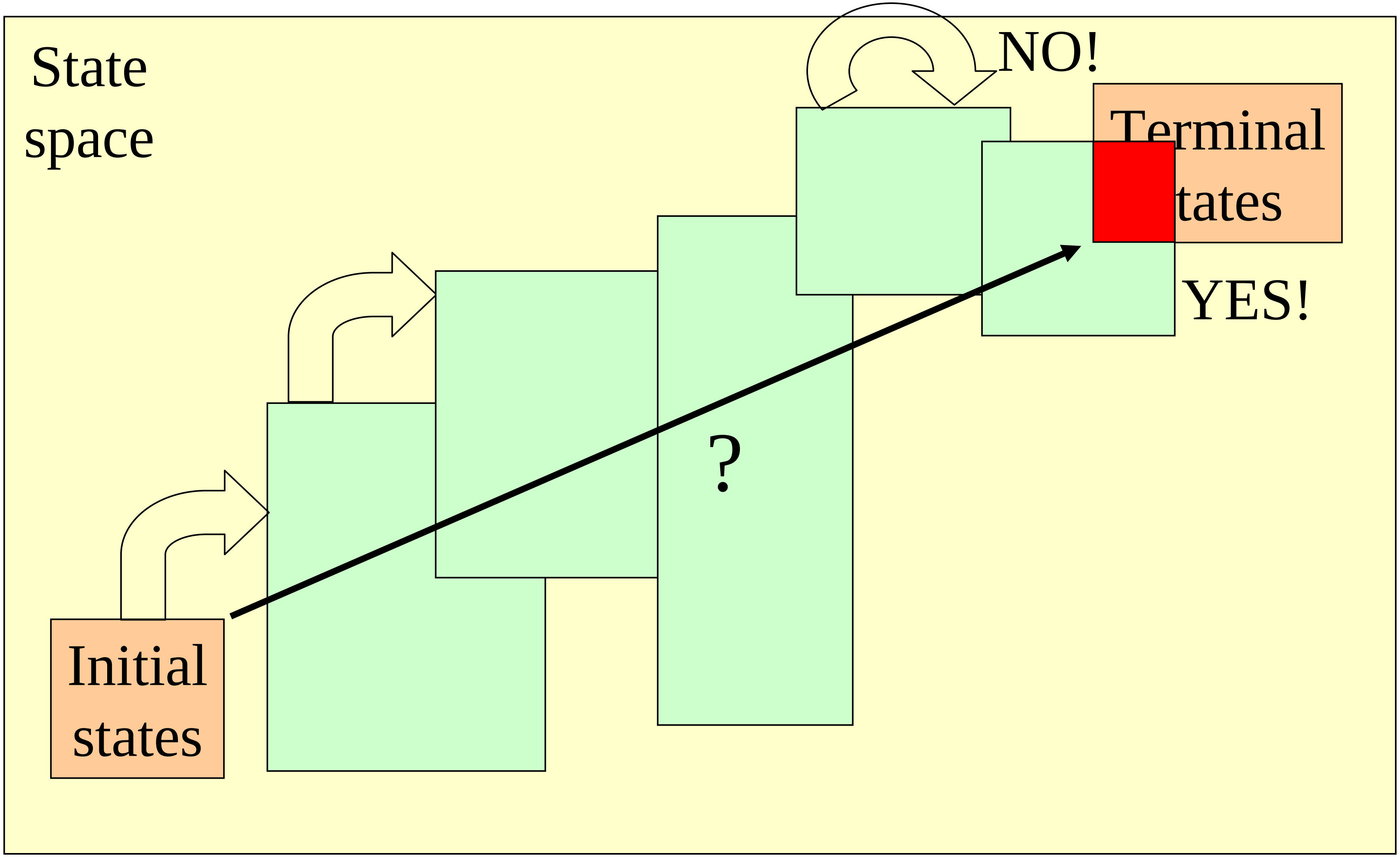
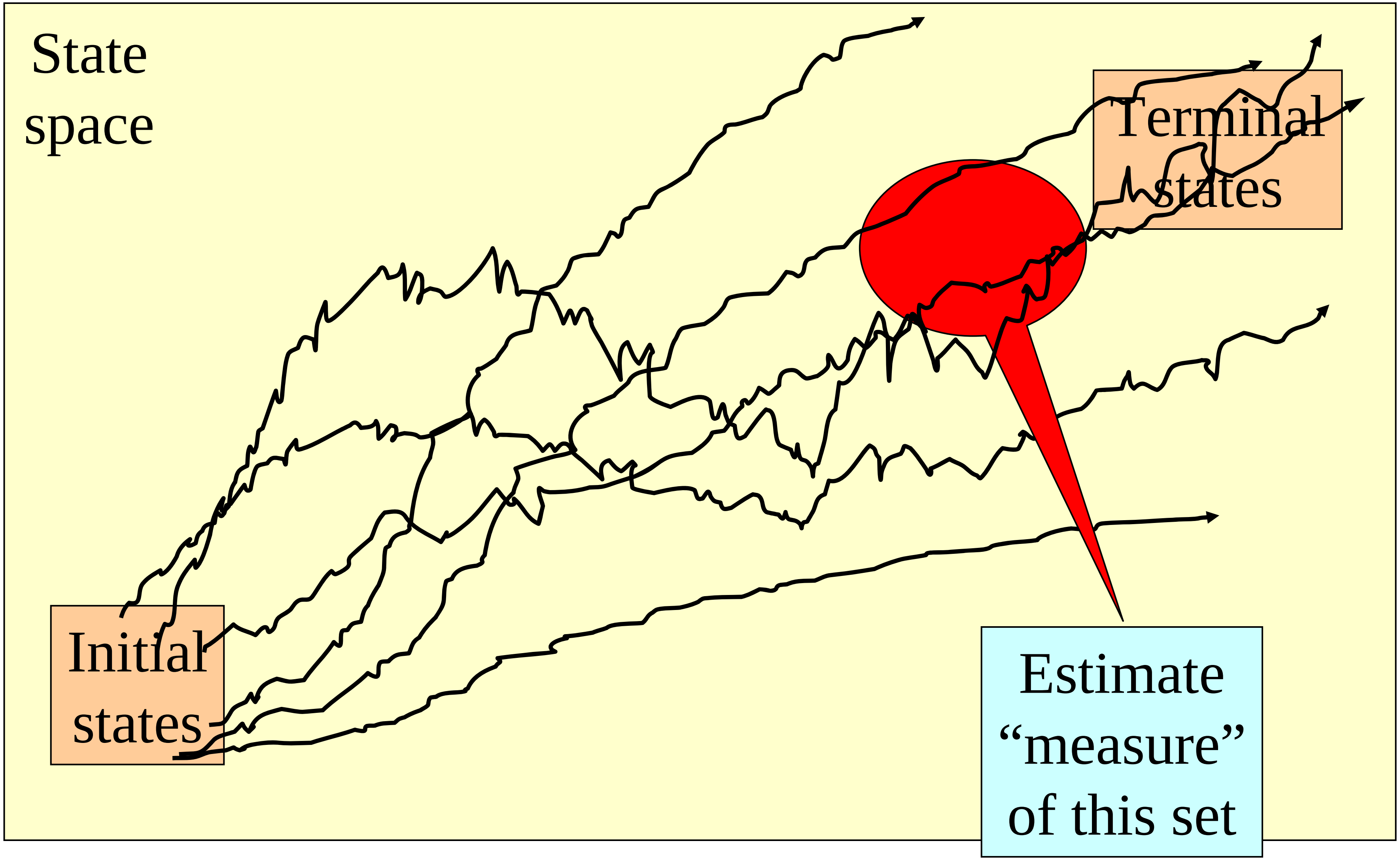
Reachability: Hybrid systems



Reachability: Stochastic HS



Reachability

- Underlying probability space (Ω, M, P)
- State space $(\mathbb{X}, B)$
- Stochastic process $x : \Omega \times \mathbb{R}_+ \to \mathbb{X}$
- Given $E \in B$ and $T \geq 0$
- Reach "events"

$$Reach_T(E) = \{\omega \in \Omega \mid \exists t \in [0, T] \text{ such that } x(\omega, t) \in E\}$$

$$Reach_\infty(E) = \{\omega \in \Omega \mid \exists t \geq 0 \text{ such that } x(\omega, t) \in E\}$$

- Reach probability

$$P\{Reach_T(E)\}$$

Alternative characterization

- Define indicator function

$$I_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

- Note that

$$I_E(x(t)) = 1 \Leftrightarrow x(t) \in E$$

$$\max_{t \in [0,T]} I_E(x(t)) = 1 \Leftrightarrow \exists t \in [0,T] : \ x(t) \in E$$

$$P\{Reach_T(E)\} = P\{\max_{t \in [0,T]} I_E(x(t)) = 1\}$$

$$= \mathbb{E}\{\max_{t \in [0,t]} I_E(x(t))\}$$

Immediate technical problem

- Is the set $Reach_T(E)$ an event?

$$Reach_T \in M$$

- Equivalently, is

$$\max_{t\in[0,T]} I_E(x(t))$$

  a random variable?
- Answer trivially yes in discrete time
- Answer is often yes in continuous time
  - E.g. SDP, PDP
- Technical conditions need to be introduced

Computation of $Reach_T$

- Analytical estimates [Bujorianu et.al.]
  - Dirichlet forms, potential theory
  - Links to Lyapunov stability
- Computational estimates [Katoen et.al.]
  - Symbolic model checking
  - Restricted classes
- Numerical estimates [Abate, Prandini, Mitchell]
  - Numerical solution of PDE's
  - Approximate by Markov chains
- Statistical estimates
  - Monte-Carlo simulation

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Monte-Carlo simulation

- Assume simulator of SHS is available
- Simulate N times $x^i(\cdot\ ) : [0, T] \rightarrow \mathbb{X}$
- Count number of simulations that reach E

$$I_{Reach_T(E)}(x(\cdot\ )) = \begin{cases} 1 & \text{if } \exists t \in [0, T] : \ x(t) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Estimate

$$\hat{P}^N = \frac{1}{N} \sum_{i=1}^{N} I_{Reach_T(E)}(x^i(\cdot\ ))$$

Convergence

- It can be shown that $\lim_{N \to \infty} \hat{P}^N = P\{Reach_T(E)\}$
- Moreover …

Probability $\left|\hat{P}^N - P\right| \geq \varepsilon$ is at most δ as long as

$$N \geq \frac{1}{2\varepsilon^2} \ln\left(\frac{2}{\delta}\right) \qquad \delta, \epsilon \in (0, 1)$$

- Simulating more we get as close as we like
- "Fast" growth with ε slow growth with δ
- No. of simulations independent of state size
- Time needed for each simulation dependent on it
- Have to give up certainty

Not as naïve as it sounds

- Efficient implementations
  - Interacting particle systems, parallelism
- With control inputs
  - Expected value cost
  - Randomized optimization problem
  - Asymptotic convergence
  - Finite sample bounds
- Parameter identification
  - Randomized optimization problem
- Can randomize deterministic problems

Particle implementation

- Each simulation treated as “particle”
- Density approximated by weighted average of particles
- Particles simulated in parallel
- Particles interact at each simulation step
- “Good” particles get rewarded
  - Assigned bigger “weight”
  - Allowed to produce more “offspring”
- Advantages
  - Substantial speed-up in rare event simulation
  - Recursive implementation

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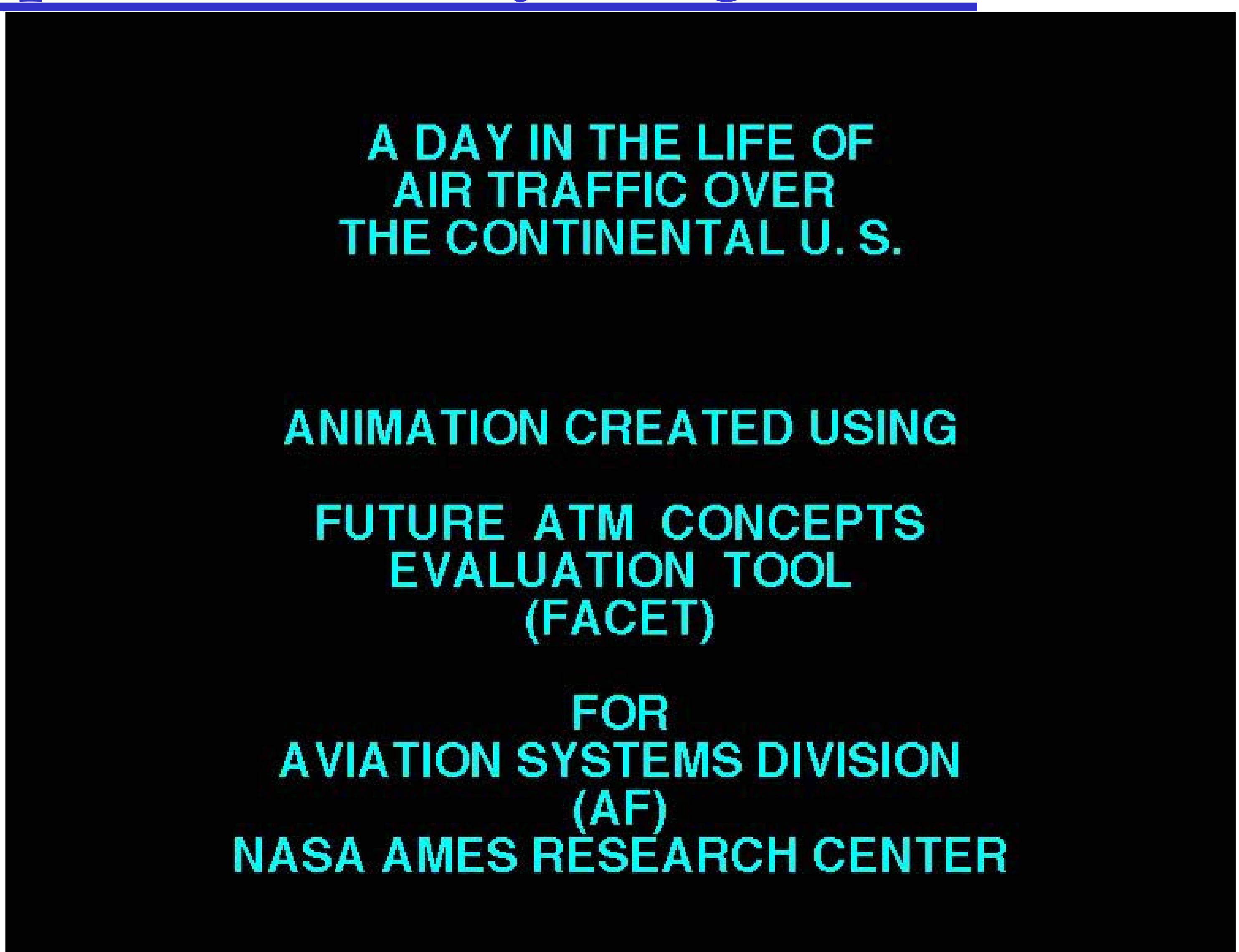
Air Traffic Management (ATM)

- Air traffic is
  - Safety critical, large scale complex system

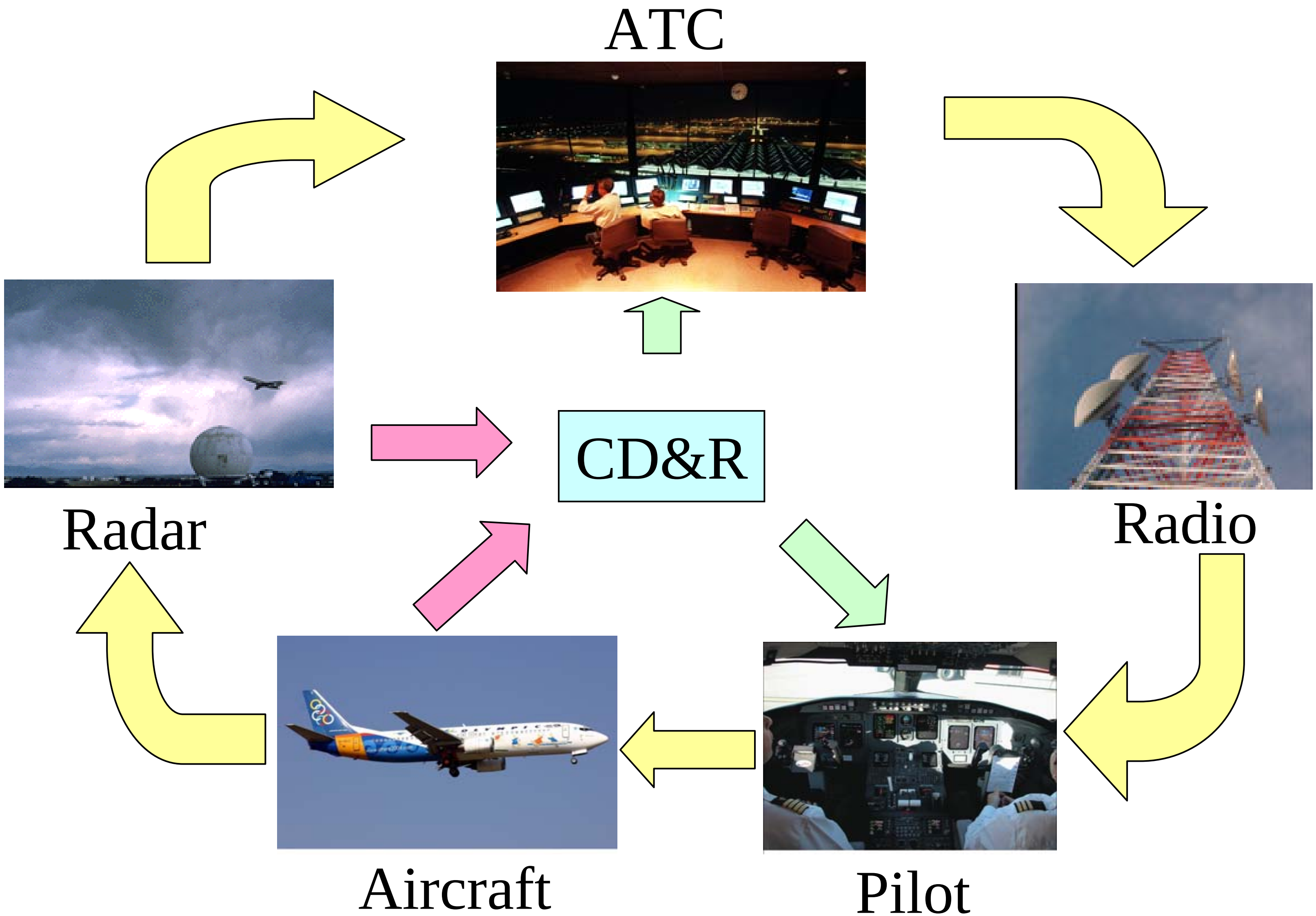
Airspace is heavily congested



Air Traffic Management (ATM)

- Air traffic is
  - Safety critical, large scale complex system
  - Economic, social, environmental constraints
  - Human in the loop system

Conflict detection and resolution (CD&R)



CD&R

- CD&R steps
  - Predict the future positions of aircraft (TP)
  - Check whether they will come close (CD)
  - Decide what to do to solve the problem (CR)
  - Tell the air traffic controller or the pilot (or not!)
    - Subliminal control
- Uncertainty
  - Wind and weather
  - Human actions
- Reachability for a stochastic hybrid system
  - CD → Reach probability estimation
  - CR → Stochastic control problem

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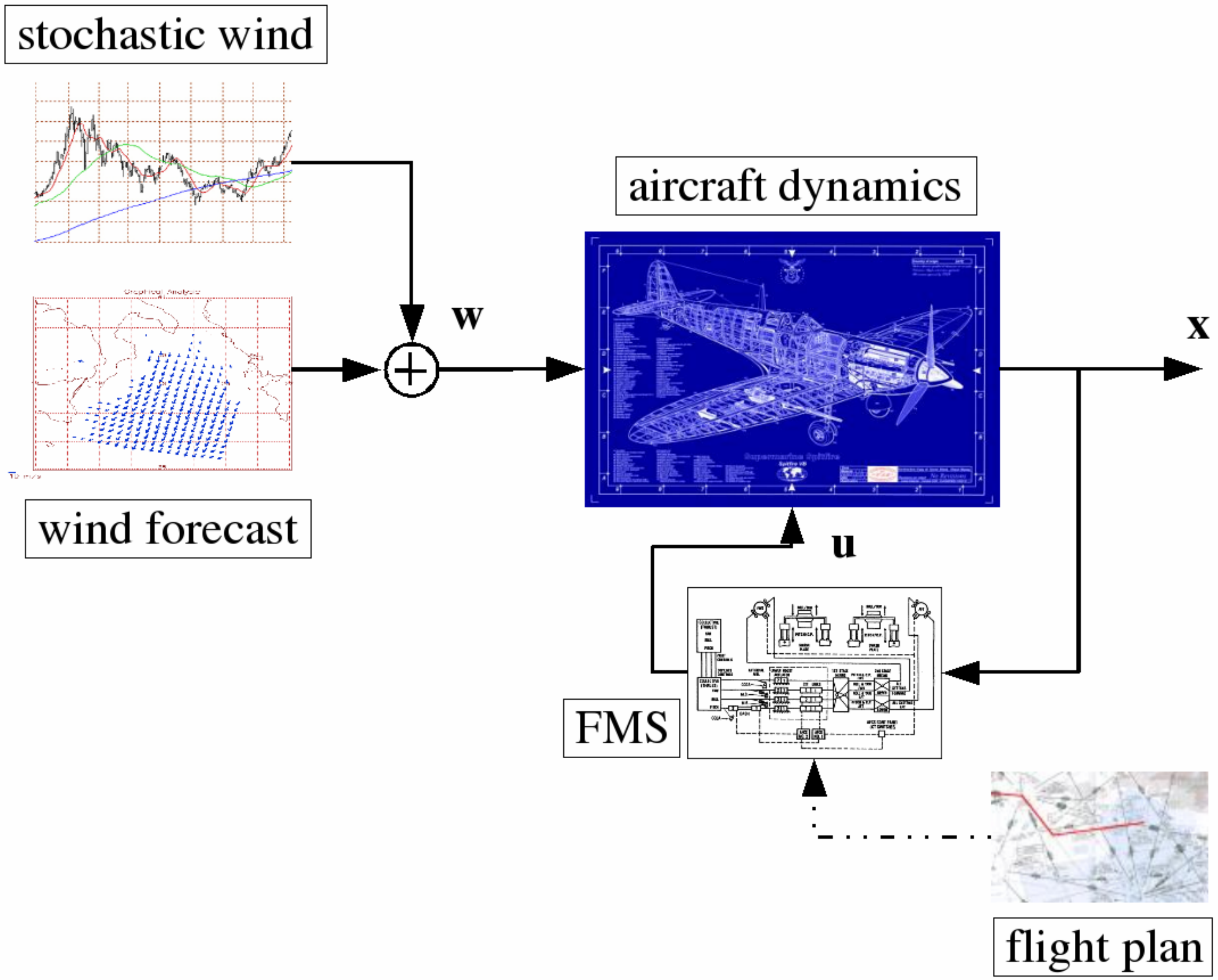
Aircraft dynamics and wind model

- Point mass model, FMS, flight plan, weather
- Hybrid & stochastic
  - Continuous: Motion of aircraft
  - Discrete: Flight plan and FMS modes
  - Stochastic: Wind uncertainty
- Multiple flights

- Different dynamics for each type of aircraft
- Includes meteorological forecasts and meteorological uncertainty

Block Diagram of the model



State of the Aircraft Model

Continuous state (x)

- Position (X,Y,h)
- True Airspeed (V)
- Heading (ψ)
- Mass (m)

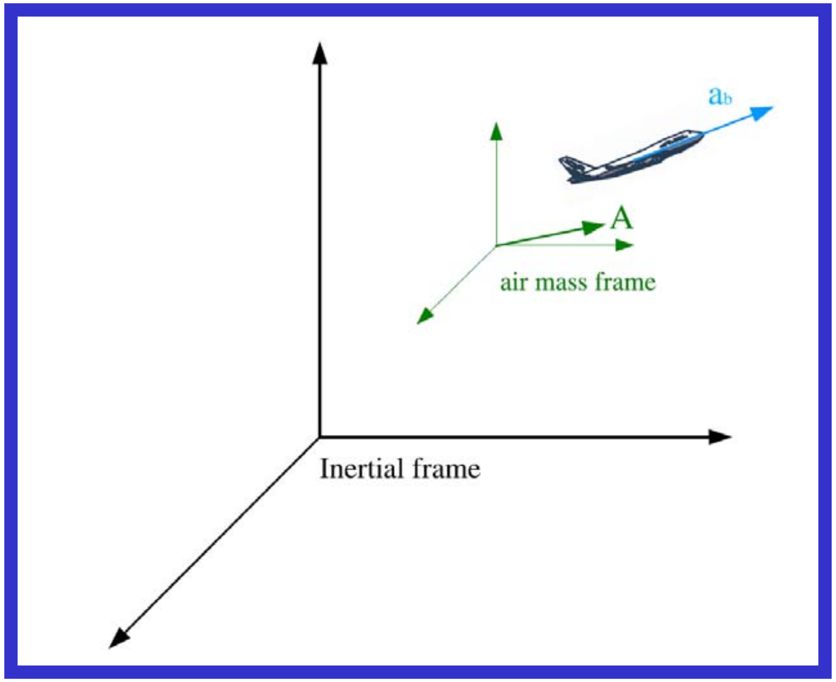
Inputs (u)

- Thrust (T)
- Bank angle (ϕ)
- Flight path (γ)

Discrete state

- FL: Flight Level
- WP: Way point index
- AM: Acceleration mode
- CM: Climb mode
- TrM: Troposphere mode
- SHM: Speed hold mode
- FP: Flight phase
- RPM: Reduced power mode
- CRM : Cruise mode

Movement in accelerating reference frame



- Aircraft accelerating with respect to air mass
- Air mass accelerating with respect to inertial frame
- Fictitious force representing latter acceleration must be included in the dynamics

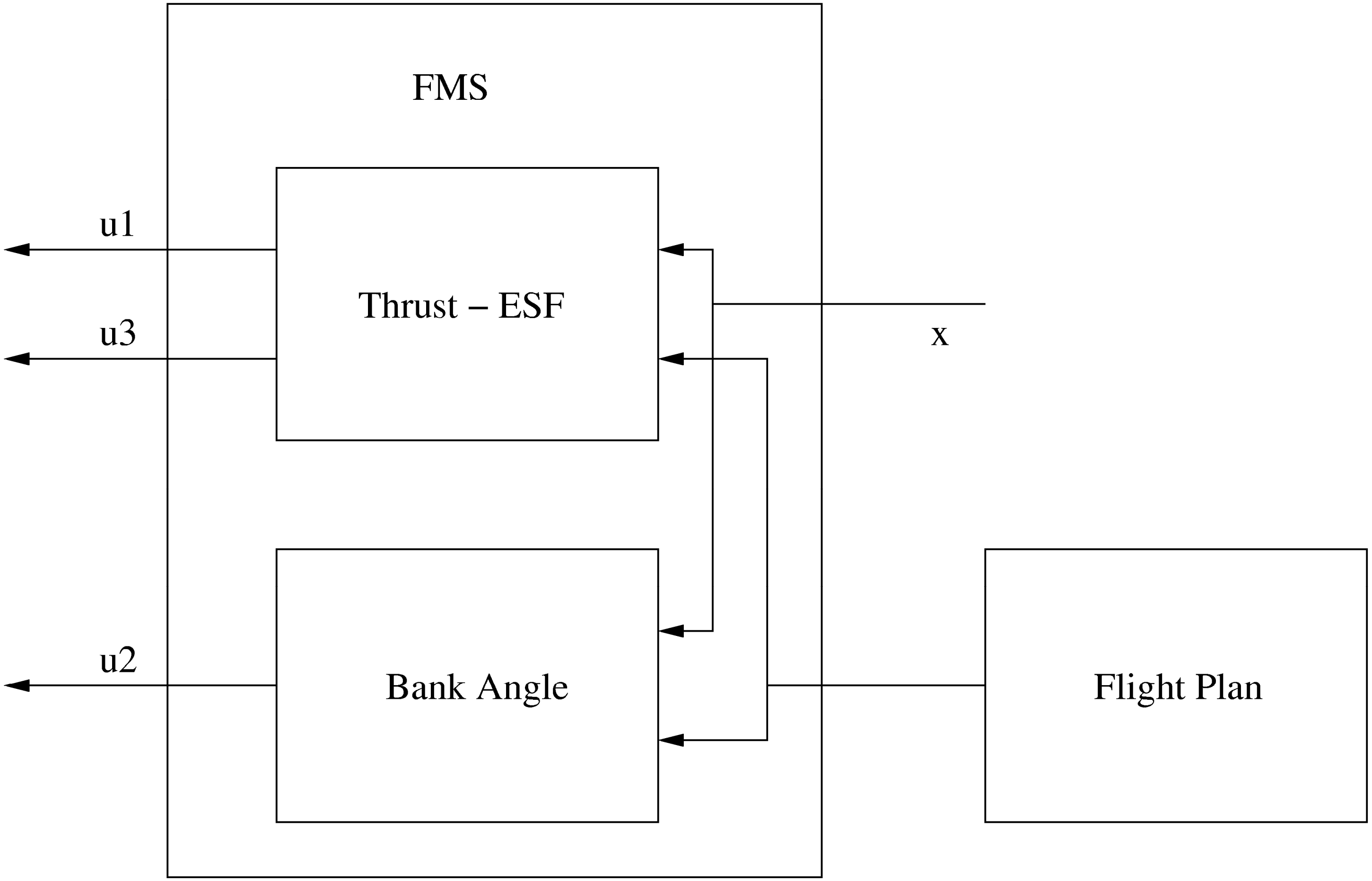
Continuous dynamics

$$\dot{x} = \begin{bmatrix} V\cos(\psi)\cos(\gamma) + w_1 \\ V\sin(\psi)\cos(\gamma) + w_2 \\ V\sin(\gamma) + w_3 \\ \frac{T-D}{m}(1-\widehat{u_3}) - W_{agf}\cos(\gamma)V\sin(\gamma) \\ \frac{L\sin(\phi)}{mV\cos(\gamma)} - W_{cgf}\tan(\gamma) \\ -\eta T \end{bmatrix}$$

Wind gradient factors

Where $x = \begin{bmatrix} X & Y & h & V & \psi & m \end{bmatrix}^T$

Flight Management System (FMS)



Hybrid FMS controllers

- Input applied for thrust, bank angle and flight path angle depends on
  - Aircraft state (feedback)
  - Flight plan (reference trajectory)
  - Internal FMS discrete state (logic)
  - Measured wind (disturbance)
- Nonlinear continuous controllers
- Logic based switching

→FMS itself a hybrid system

Stochastic terms

- Wind consists of two parts
  - Nominal (meteorological forecast, assumed available)
  - Stochastic (uncertainty of the forecast)
- Stochastic part correlated in time & space

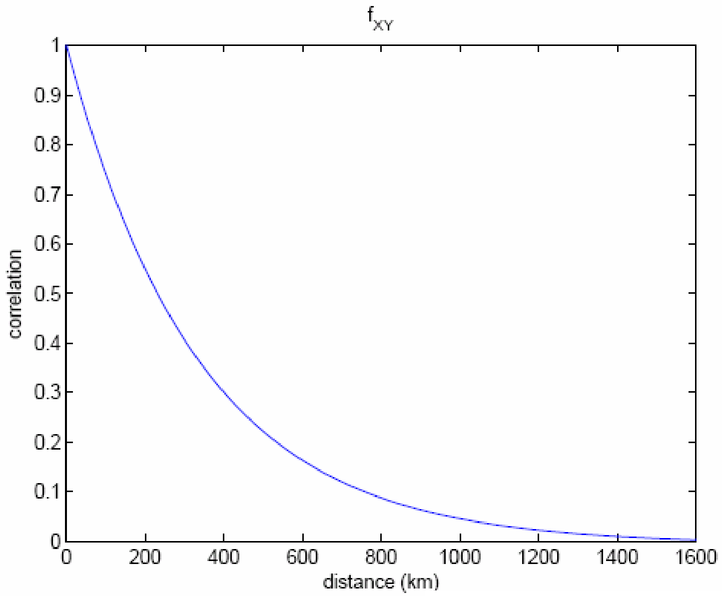
$$r_{xy}\left(t, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, t', \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}\right) = \sigma(x_3)\sigma(x_3') f_t(|t-t'|) f_{x_1 x_2}\left(\left\| \begin{matrix} x_1 - x_1' \\ x_2 - x_2' \end{matrix} \right\|\right) f_{x_3}\left(|p(x_3) - p(x_3')|\right).$$

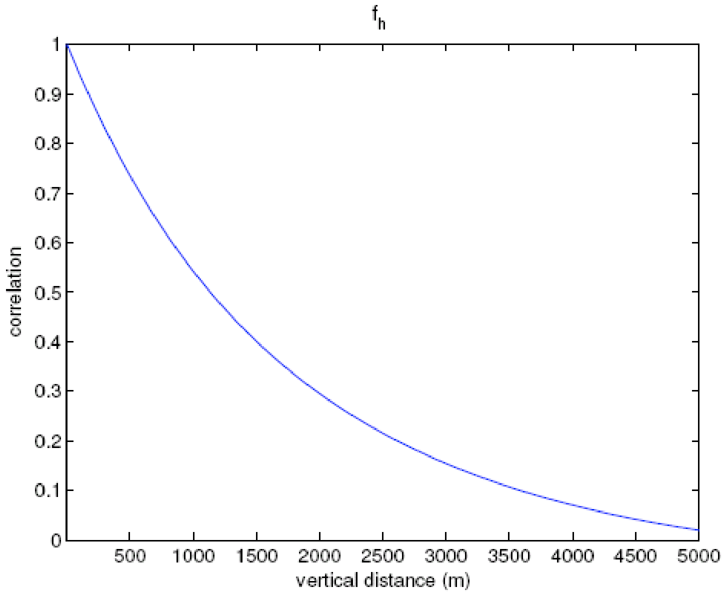
- Time correlation

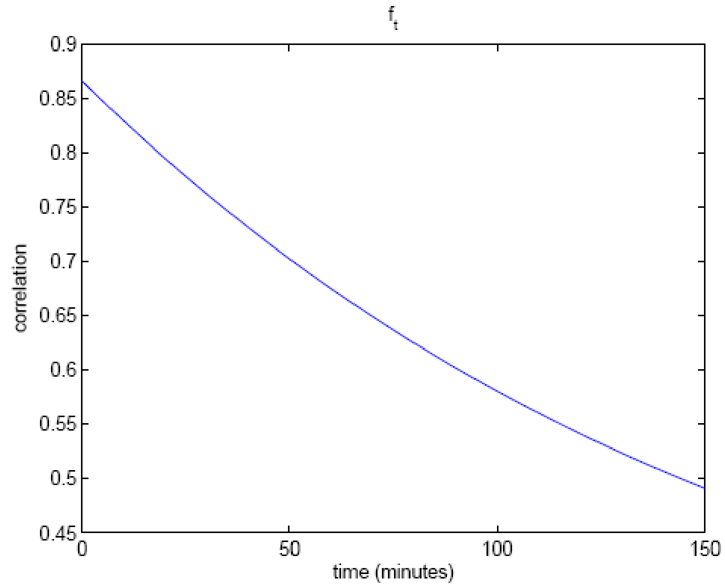
$$f_t(\chi) = c_t + (1 - c_t - d_t)e^{-\frac{\chi}{b_t}} + d_t \cos\left(2\pi \frac{\chi - e_t}{g_t}\right)$$

- Horizontal correlation $f_{x_1 x_2}(\chi) = c_{xy} + (1 - c_{xy})e^{-\frac{\chi}{b_{xy}}}$
- Altitude correlation $f_{x_3}(\chi) = c_z + (1 - c_z)e^{-\frac{\chi}{b_z}}$

Wind uncertainty correlation







Stochastic terms

- Even extracting random variable with appropriate correlation is hard!
  - Only at aircraft positions → accurate in level flight
  - On 4D grid + quadrilinear interpolation → better for climbing and descending aircraft
  - Linear filter driven by noise → approximate but more useful for adaptive trajectory prediction
- Analytical and numerical solutions hopeless
- Use Monte Carlo methods
  - Adaptive trajectory prediction
  - Conflict detection
  - Conflict resolution

Example

Heathrow
0:45-10:00am
November 7, 2002

Courtesy of I. Lymperopoulos

Outline

1. Modeling
   1. Classes of models
   2. Comparison
2. Analysis and control
   1. Overview of analysis and control problems
   2. Reachability
   3. Monte-Carlo approximation
3. Air traffic control
   1. Overview, CD&R
   2. Stochastic hybrid modeling
   3. Adaptive trajectory prediction
   4. Optimal control
4. Concluding remarks & references

Trajectory Prediction

- Use radar measurements to estimate future aircraft positions
  - Hence determine safety of situation
- Unknown parameter values reduce accuracy
  - Aircraft mass
  - Wind error (real wind – forecasted wind)
- Use radar measurements to estimate values
- Parameter space of 161 dimensions
  - Wind changes only with altitude.
  - Considered constant in time and horizontal space.

State estimation problem

- Discrete time, sampled at radar sweeps
- Append parameters to aircraft state
- Consider the evolution of the state sequence $\{x_k, k \in \mathbb{N}\}$ given by

$$x_k = f_k(x_{k-1}, v_{k-1})$$

i.i.d. process noise

- Recursively estimate of x_k from noisy radar measurements

$$z_k = h_k(x_k, n_k)$$

i.i.d. measurement noise

- Use estimate + model to predict future

Employing particle filters

- Functions of state process

  $$f_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \longrightarrow \mathbb{R}^{n_x}$$

  and measurement

  $$h_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_n} \longrightarrow \mathbb{R}^{n_z}$$

  are non-linear
- The model too complex for analytical or numerical approximation
- Multiple targets
- Multiple sensors
- Use particle filters to approximate statistically

Particle Filters (Sequential Monte Carlo)

- Estimate of complete probability density function (pdf)
- Sequential importance sampling
  - Maintain particle population (parameter values)
  - Each with a weight (likelihood of being correct)
- Approximate pdf by weighted average
- Approximation of pdf instead of model
- Features
  - General, scalable, parallelizable
  - Theoretical convergence guarantees
  - Probabilistic finite sample bounds
- Potential problems
  - Degeneracy
  - Diffusion

SIS - Algorithm

1. Generate N particles (weather scenarios and aircraft mass) according to some available prior information
2. Use the process function $f_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \longrightarrow \mathbb{R}^{n_x}$ to evolve the particle in the next state

$$x_k = f_k(x_{k-1}, v_{k-1})$$

3. Use the measurement function $h_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_n} \longrightarrow \mathbb{R}^{n_z}$ to calculate the likelihood of each particle
4. Assign weights to particles proportional to likelihoods
5. Return to step (2)

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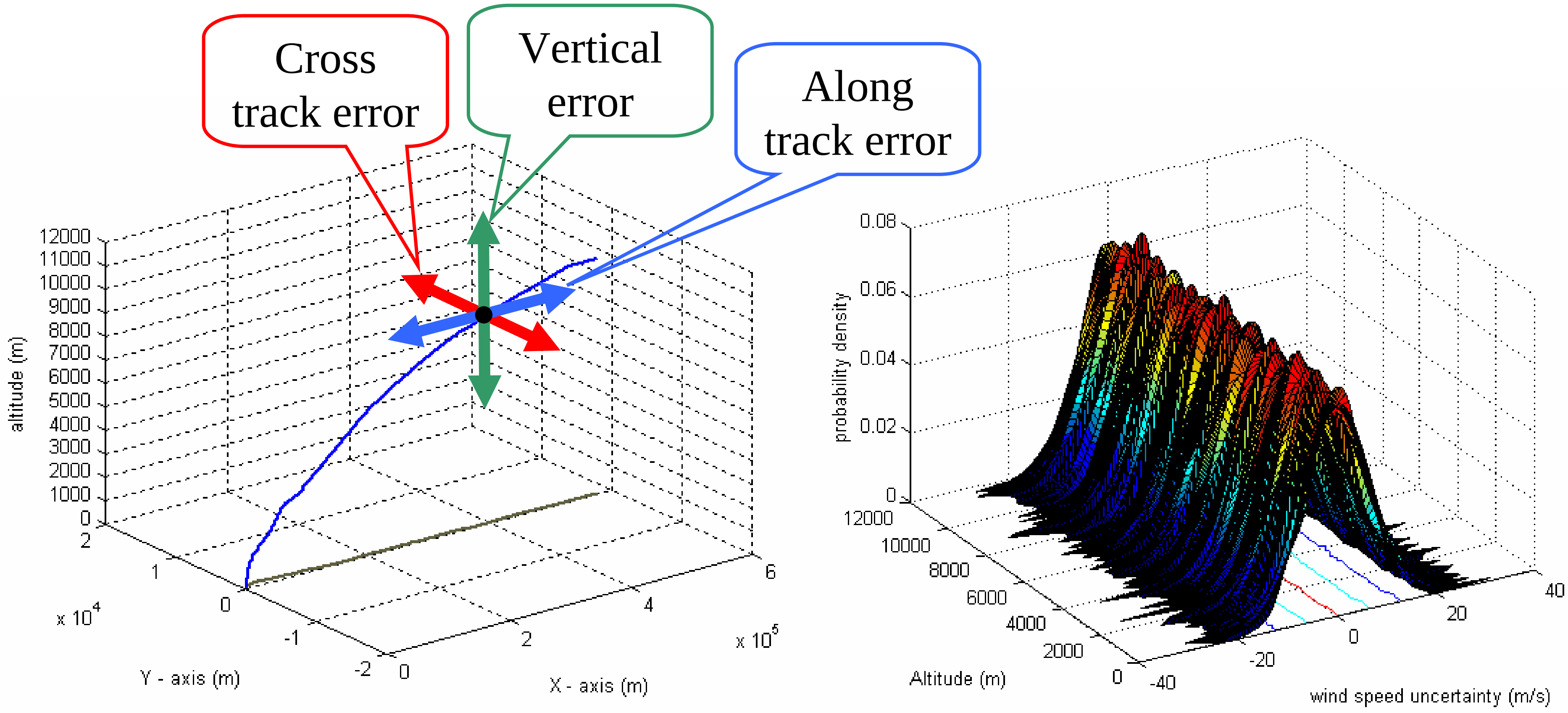
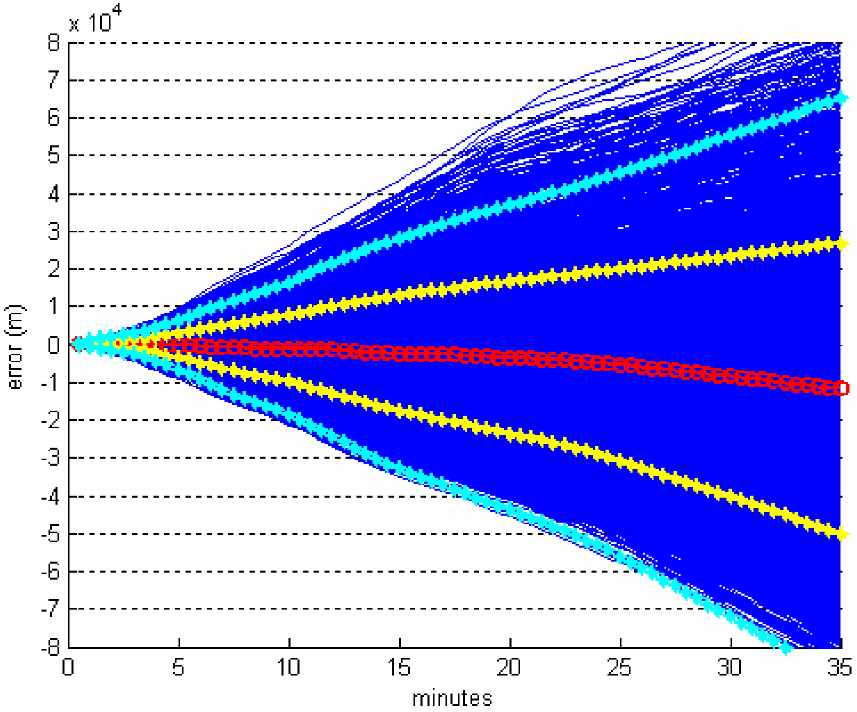
4. Assign weights to particles proportional to likelihoods
5. Sample N new particles from weighted posterior distribution
6. Return to step (2)

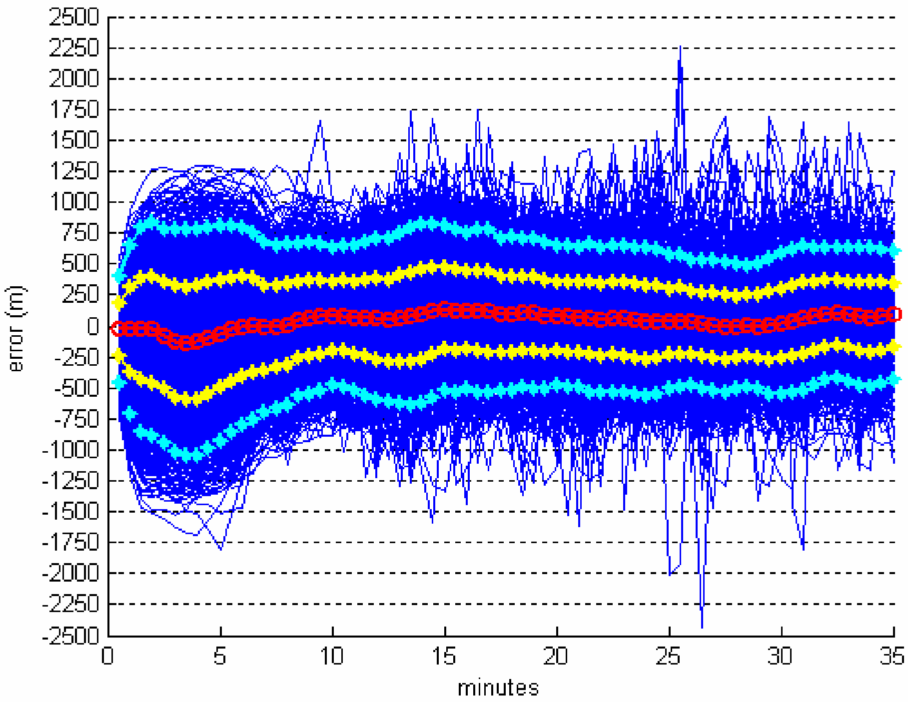
Results: Aircraft at take-off



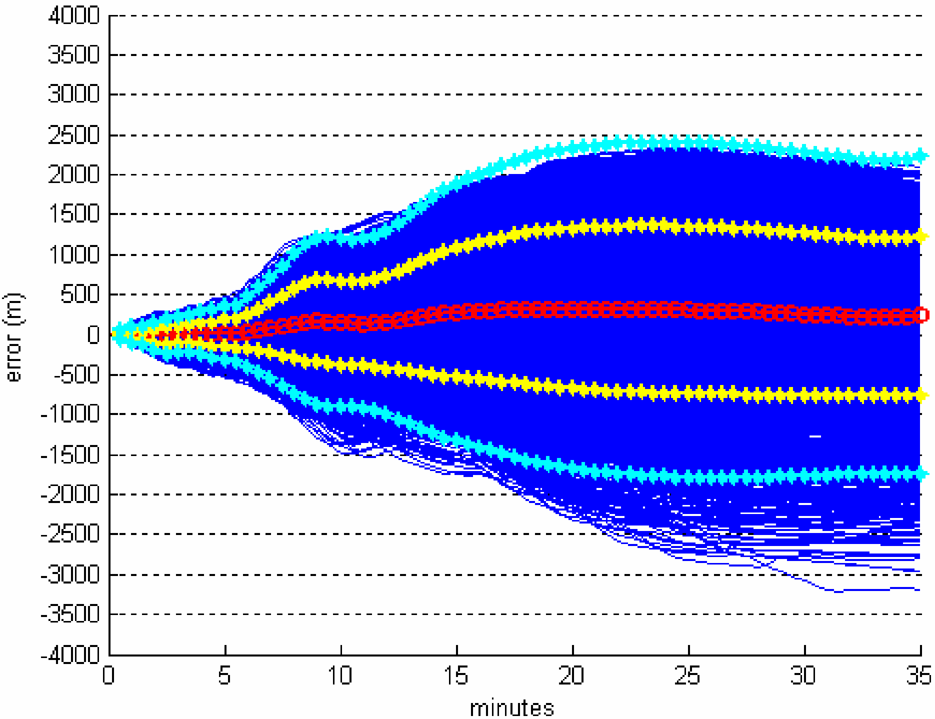
Radar measurements every 30 seconds

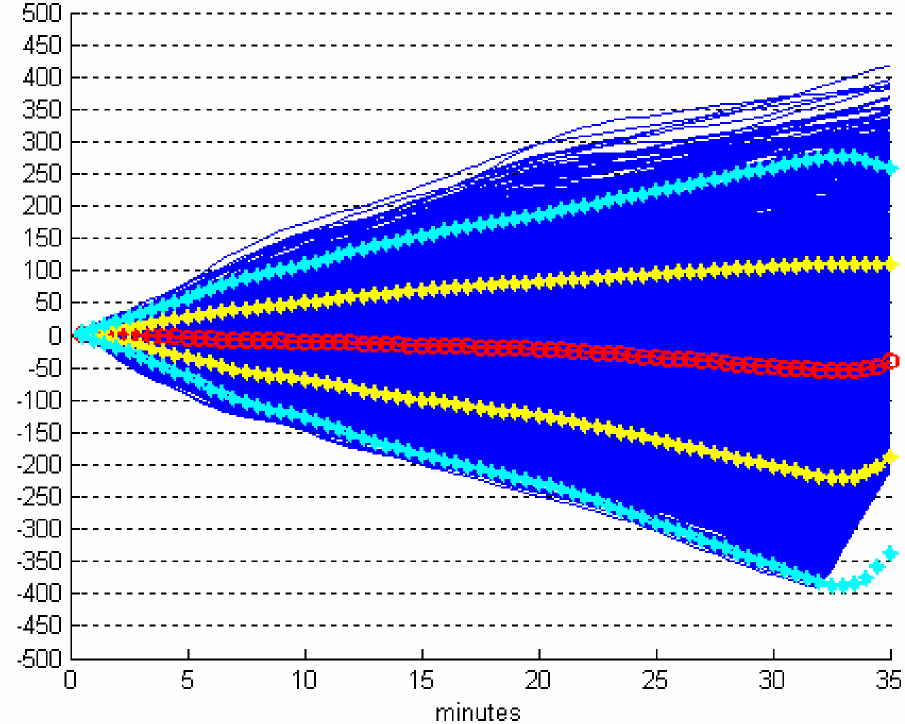
Initial uncertainty along & cross track



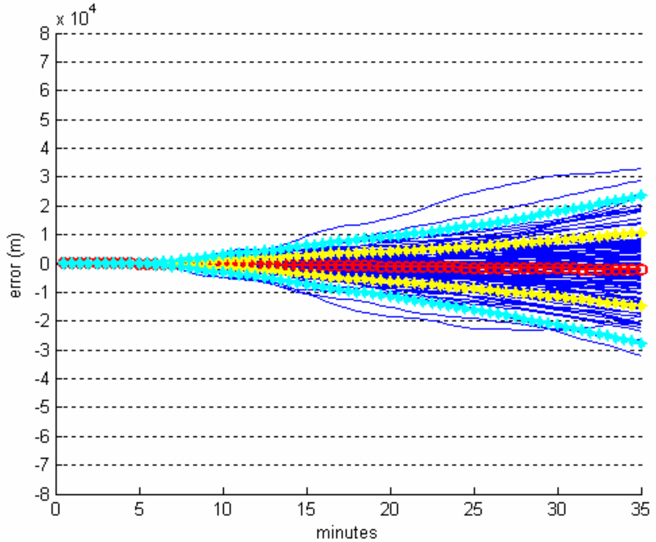


Initial uncertainty vertical and time

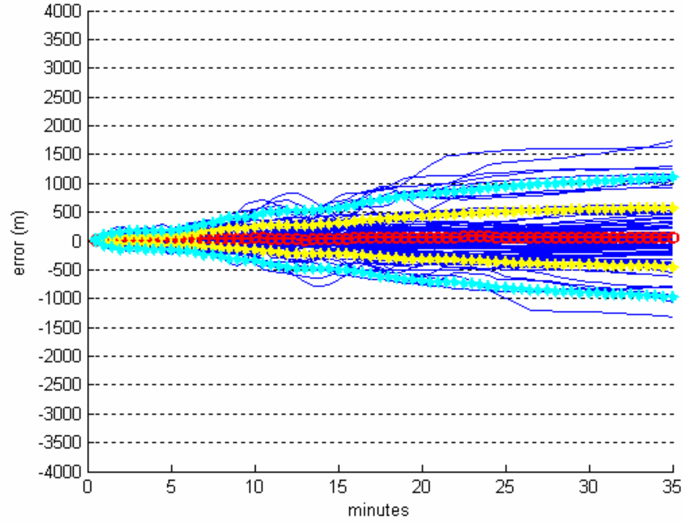


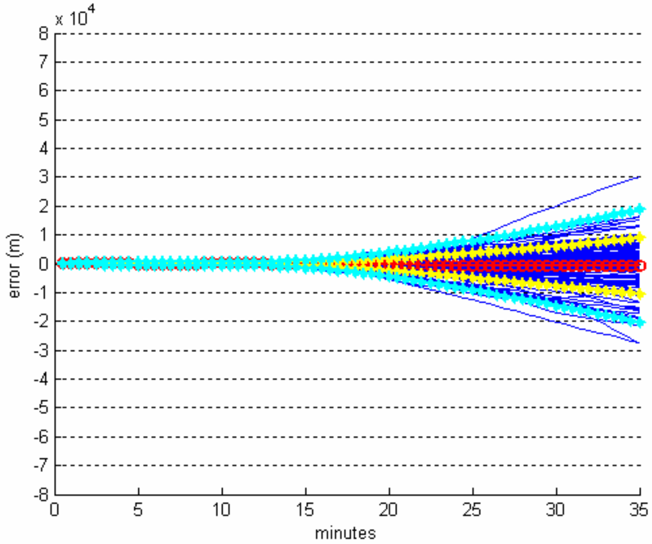


Uncertainty reduction

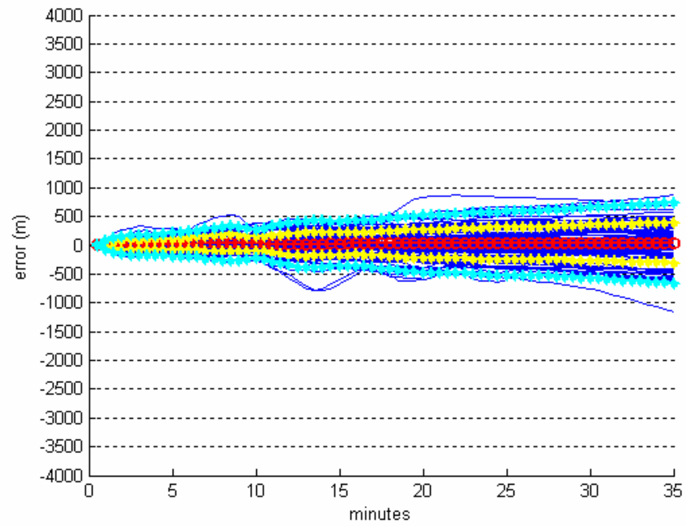


10 measurements





30 measurements



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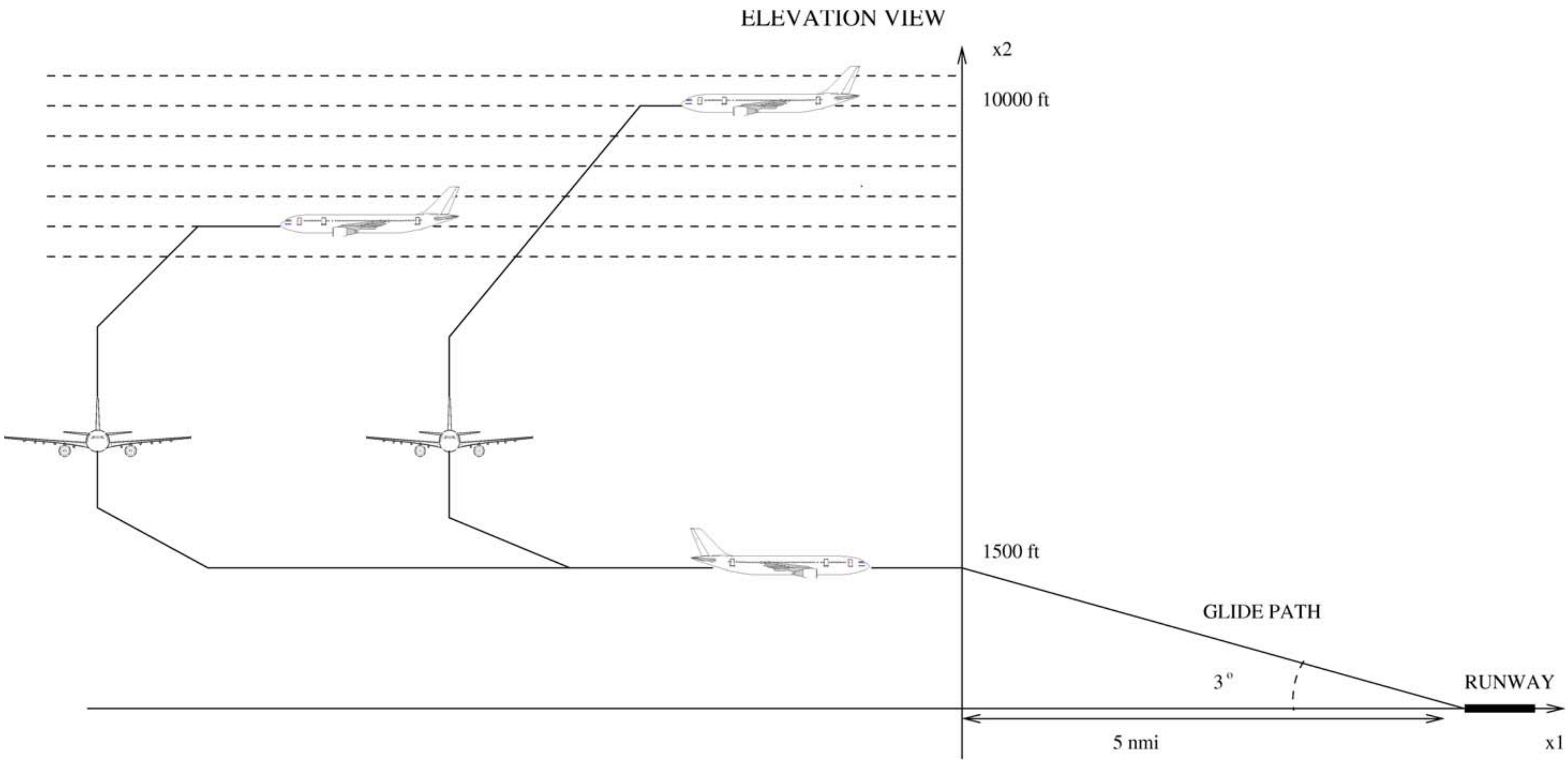
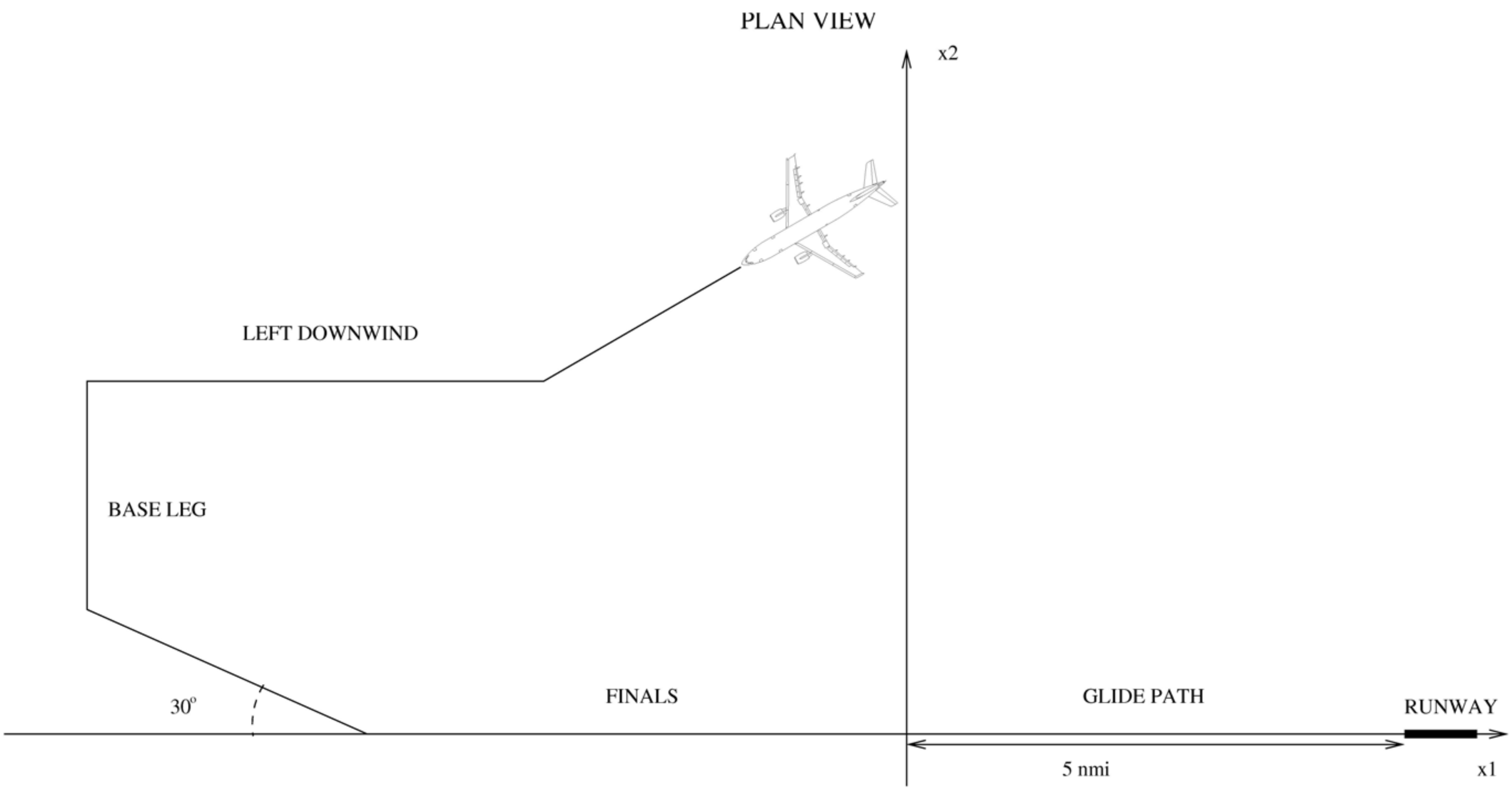
Final approach: Side view

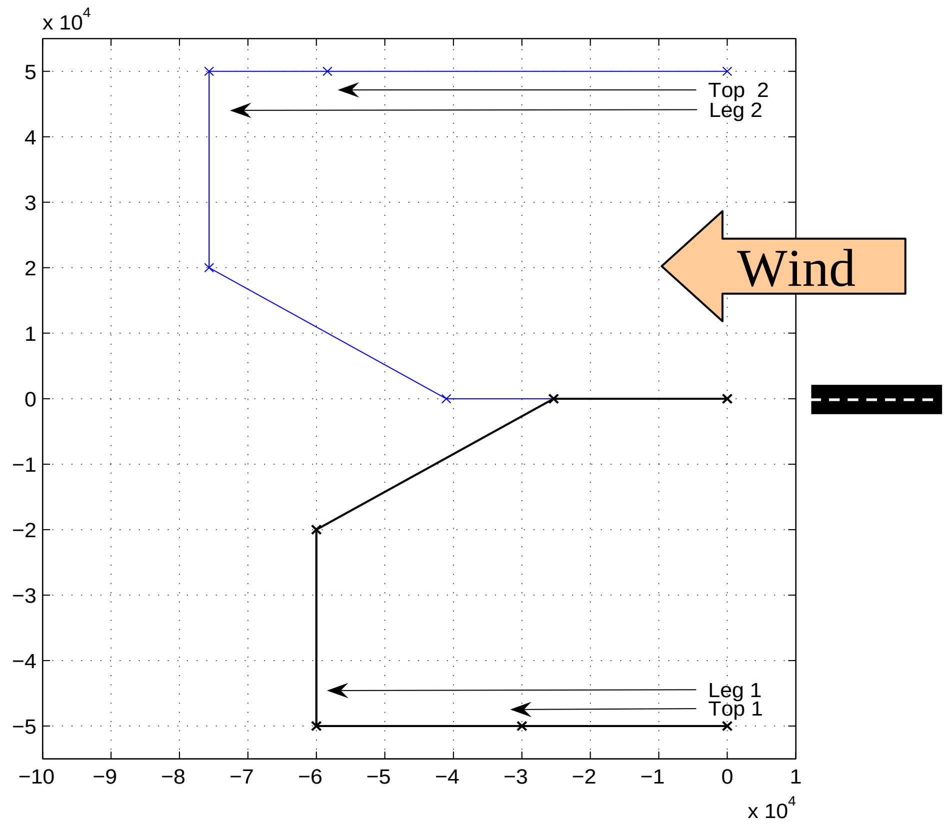


Final approach: Top view



Sequencing for final approach

- Two aircraft A1, A2
- Come in at FL150
- A1 flight plan fixed
- Optimize A2 approach
  - Collision free
  - Reach FL15 at end
  - Minimum time
- A2 control parameters
  - Top of descent
  - Length of downwind leg
- Probabilistic mass and wind uncertainty



Optimization with expected value criterion

- Consider domain $\Omega \subseteq \mathbb{R}^n$, reward function

$$U : \Omega \to \mathbb{R}$$

- Interested in maximization problem

$$U^* = \sup_{\omega \in \Omega} U(\omega) \in \mathbb{R}, \quad \bar{U} = \left\{ \omega \in \Omega \,|\, U(\omega) = U^* \right\} \subseteq \Omega$$

- Stochasticity → reward an expected value
- Random variable $x \in X$

$$\left. \begin{array}{l} x \sim p_\omega(x) \\ u(\bullet,\bullet) : U \times X \to \mathbb{R} \end{array} \right\} \to \boxed{U(\omega) = \int_X u(\omega, x)\, p_\omega(dx)}$$

Optimization problem

- Optimization parameters
  - Position of top of descent, length of downwind leg
  - Domain $\Omega = \{(\omega_1, \omega_2) \mid \omega_2 \in [35k, 60k], \omega_1 \in [0, \omega_2]\}$
- Random variable x
  - State of two aircraft
  - Sampled every radar sweep
- Distribution $p_\omega(x)$ generated by simulator
- Reward function
  - Land as quickly as possible $\text{perf}(\omega, x) = e^{-aT_2}$
  - Avoid collisions
  - Get to FL15

Reachability constraint

Landing time of A2

Encoding reachability constraints

- Reachability constraints define "safe" set $X_f \subseteq X$
- Would like to optimize subject to $x \in X_f$
- Next best thing: Lower bound the probability that $x \in X_f$ for the optimal solution
- Discontinuous "barrier function"
- If $\mathrm{perf}(\omega,\mathrm{x}) \in [0,1]$

$$u(\omega, x) = \begin{cases} \mathrm{perf}(\omega, x) + \Lambda & x \in X_f \\ 1 & x \notin X_f \end{cases}$$

Proposition: If an almost surely safe ω exists then

$$\hat{\omega} \in \bar{U} \Rightarrow \int_{x \notin X_f} p_{\hat{\omega}}(dx) \leq \frac{1}{\Lambda}$$

Metropolis-Hastings algorithm

- Related to simulated annealing
- Expected value cost
- Finite sample bounds
- Roughly speaking
  - Select resolution maneuver according to probability distribution
  - Estimate cost by running simulations
  - "Accept" maneuver with certain probability that depends on whether it is better than previous
  - Probability distribution of "accepted" maneuvers will converge around optimal

Metropolis-Hastings algorithm

Algorithm:

1. Set $k=0$, extract $\omega_0 \in \Omega$
2. Simulate $X_1,\dots X_J \sim p_{\omega_0}(x)$ compute $U_0 = \prod_{i=1}^{J} u(\omega_0, X_i)$
3. Extract $\hat{\omega} \sim g(\omega \mid \omega_k)$
4. Simulate $X_1,\dots X_J \sim p_{\hat{\omega}}(x)$ compute $\hat{U} = \prod_{i=1}^{J} u(\hat{\omega}, X_i)$
5. Set $\rho = \min\left\{1, \frac{g(\omega_k \mid \hat{\omega})\hat{U}}{g(\hat{\omega} \mid \omega_k)U_k}\right\}$
6. Set $(\omega_{k+1}, U_{k+1}) = \begin{cases} (\hat{\omega}, \hat{U}) & \text{w.p. } \rho \\ (\omega_k, U_k) & \text{otherwise} \end{cases}$
7. Set $k \rightarrow k+1$ and return to 3

The idea

- Construct Markov Chain over Ω
- With stationary distribution

$$\pi(d\omega) \propto \left[U(\omega)\right]^J \pi_{\text{Leb}}(d\omega)$$

- Simulate Markov Chain
- Simulation "quickly" converges to stationary distribution (conditions)
- Soon we start extracting from desired distribution

Stationary distribution (Muller et.al. 2004)

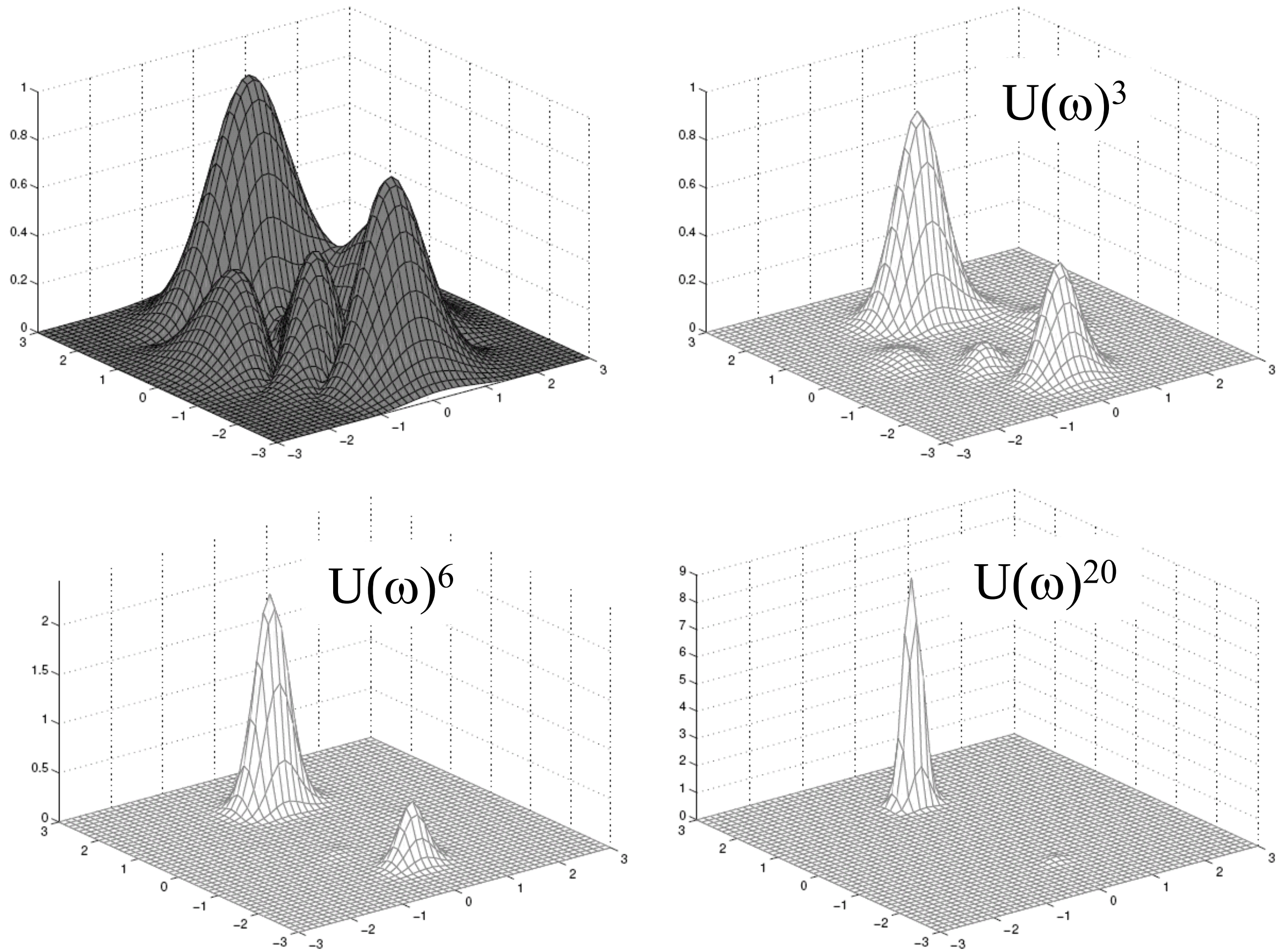
- Algorithm parameters:
  - "Proposal distribution", *g*
  - Number of simulations, *J*
- Used together as in simulated annealing
- Stationary distribution

$$h(\omega, x_1, x_2, \ldots, x_J) \propto \prod_{i=1}^{J} u(\omega, x_i)\, p_\omega(x_i)$$

- Marginal distribution

$$H(\omega) \propto \left[\int_X u(\omega, x)\, p_\omega(dx) \right]^J = \left[U(\omega) \right]^J$$

Power play



Convergence to stationary distribution

- Let $\omega_k \sim P_k(\omega)$
- Assume $g(\omega \mid \omega_k)$ independent of ω_k and

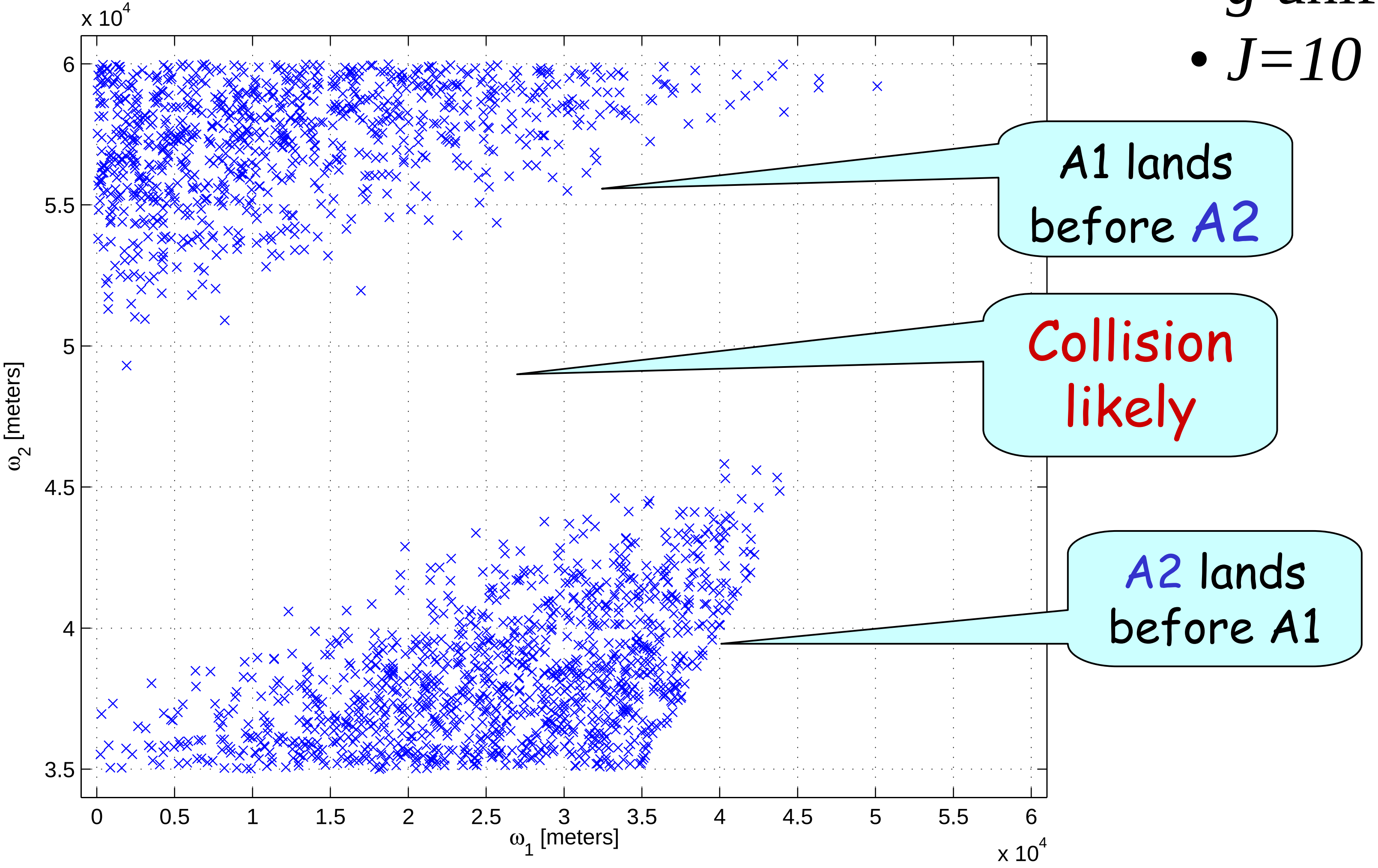
$$H(\omega) \leq Mg(\omega), \quad M > 1$$

"**Theorem**": (Meyn & Tweedy 1993)

$$\left\| P_k(\omega) - H(\omega) \right\|_{TV} \leq 2\left(1 - \frac{1}{M}\right)^k$$

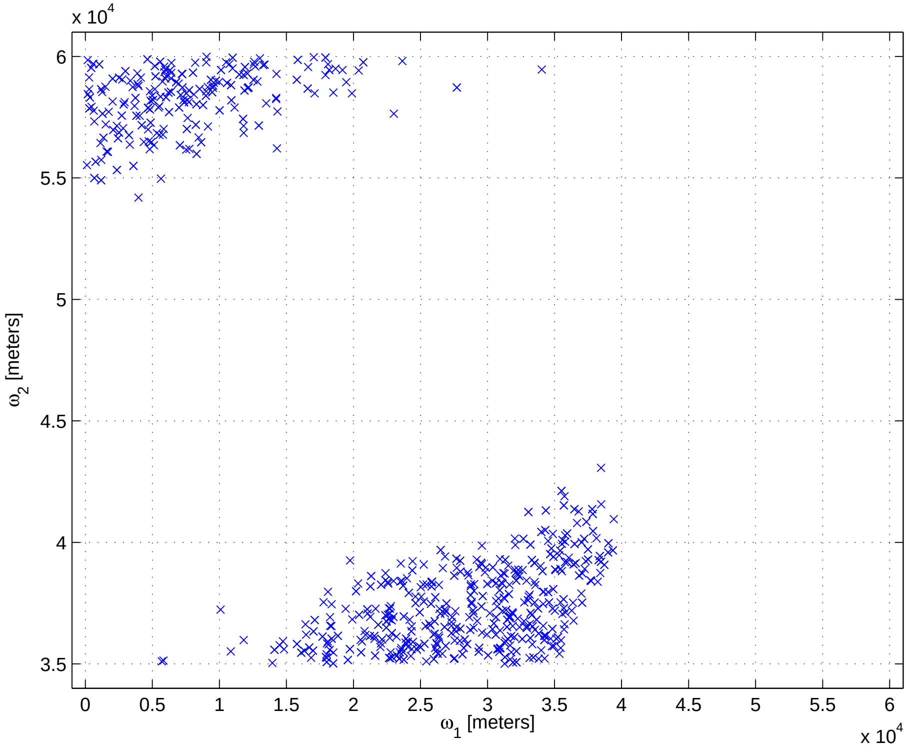
Total variation norm

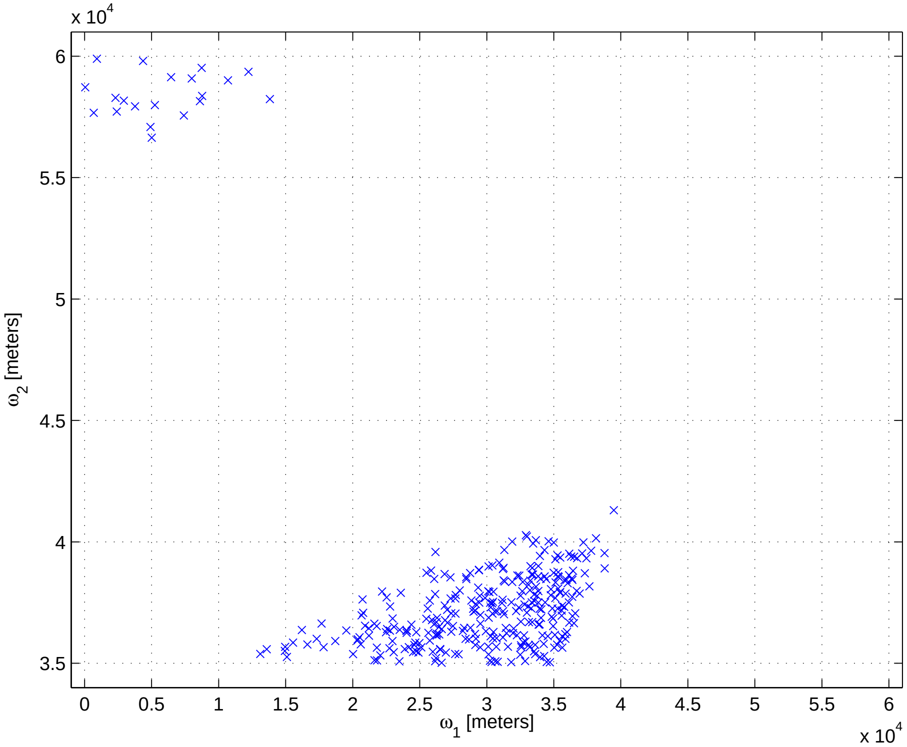
Optimization results



- *g* uniform
- *J=10*

Optimization results





- g: $N(\mu,\sigma^2 I)$, 100 μ from $J=10$
- $J=50$

- g: $N(\mu,\sigma^2 I)$, 100 μ from $J=50$
- $J=100$

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Concluding remarks

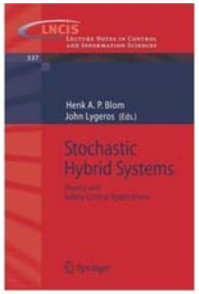
- SHS different mixtures of discrete, continuous and probabilistic terms
- Modeling of SHS rich and varied
- Foundation for
  - Theoretical contributions (e.g. optimal control, reachability, stability)
  - Computational tools (e.g. Monte-Carlo simulation)
- Technically very challenging
- Application driven
- Applications push for theoretical extensions
  - E.g. spatio-temporal correlation of wind in ATM

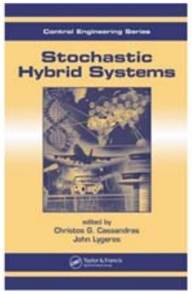
Shameless plug

- Edited volumes

  H.A.P. Blom and J. Lygeros (eds.), *"Stochastic hybrid systems: Theory and safety critical applications"*, Springer-Verlag, 2006

  C.G. Cassandras and J. Lygeros (eds.), *"Stochastic hybrid systems"*, CRC Press, 2006





- Applications to systems biology

  HYGEIA PhD school on hybrid systems biology
  Siena, Italy, July 20, 2007
  Organizers: G. Ferrari Trecate and J. Lygeros

References

1. A. Bensoussan and J.L. Menaldi, "Stochastic hybrid control", *Journal of Mathematical Analysis and Applications*, 249:261-288, 2000.
2. M. Bujorianu, "Extended stochastic hybrid systems and their reachability problem", in R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control*, number 2993 in LNCS, pages 234-249. Springer-Verlag, 2004.
3. M.H.A. Davis, "Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models", *Journal of the Royal Statistical Society, B*, 46(3):353-388, 1984.
4. M.H.A. Davis, *Markov Processes and Optimization*, Chapman & Hall, London, 1993.
5. M.K. Ghosh, A. Arapostathis, and S.I. Marcus, "Optimal control of switching diffusions with application to flexible manufacturing systems", *SIAM Journal on Control Optimization*, 31(5):1183-1204, September 1993.
6. M.K. Ghosh, A. Arapostathis, and S.I. Marcus, "Ergodic control of switching diffusions", *SIAM Journal on Control Optimization*, 35(6):1952-1988, November 1997.
7. J. Hespanha. "Stochastic hybrid systems: Application to communication networks", in R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control*, number 2993 in LNCS, pages 387-401. Springer-Verlag, 2004.
8. W. Glover and J. Lygeros, "A stochastic hybrid model for air traffic control simulation," in *Hybrid Systems: Computation and Control* (R. Alur and G. Pappas, eds.), no. 2993 in LNCS, pp. 372–386, Springer Verlag, 2004
9. I. Limperopoulos, J. Lygeros, and A. Lecchini, "Model based aircraft trajectory prediction during takeoff," in *AIAA Conference on Guidange Navigation and Control*, Keystone, Colorado, USA, August 21-24 2006

References

10. J. Hu, J. Lygeros, and S.S. Sastry, “Towards a theory of stochastic hybrid systems”, in Nancy Lynch and Bruce H. Krogh, editors, *Hybrid Systems: Computation and Control*, number 1790 in LNCS, pages 160-173, Springer-Verlag, Berlin, 2000.
11. X. Mao, “Stability of stochastic differential equations with Markovian switching”, *Stochastic Processes and Applications*, 79:45-67, 1999.
12. J.L. Menaldi, “Stochastic hybrid optimal control models”, *Aportaciones Matematicas*, 16:205-250, 2001.
13. C. Yuan and X. Mao, “Asymptotic stability in distribution of stochastic differential equations with Markovian switching”, *Stochastic Processes and Applications*, 103:277-291, 2003.
14. M. Bujorianu and J. Lygeros, “Toward a general theory of stochastic hybrid systems,” in *Stochastic Hybrid Systems: Theory and Safety Critical Applications* (H. Blom and J. Lygeros, eds.), LNCIS 337, 3–30, Springer-Verlag, 2006
15. G. Pola, M.L. Bujorianu, J. Lygeros and M. Di Benedetto, “Stochastic Hybrid Models: An Overview with Applications to Air Traffic Management”, in proceedings of *IFAC Conference on Analysis and Design of Hybrid Systems (ADHS03)*, Saint Malo, France, June 16-18, 2003
16. A. Lecchini, W. Glover, J. Lygeros, and J. Maciejowski, “Monte Carlo optimization for conflict resolution in air traffic control,” *IEEE Transactions on Intelligent Transportation Systems*, 7:470–482, 2006