Observability and State Estimation for Hybrid Systems

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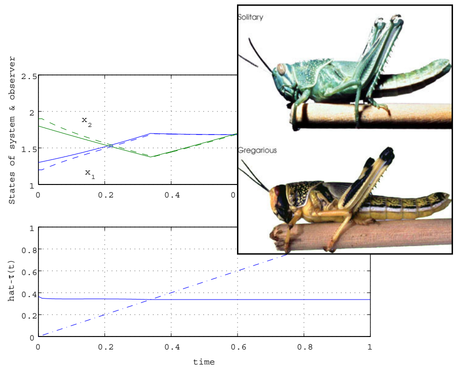
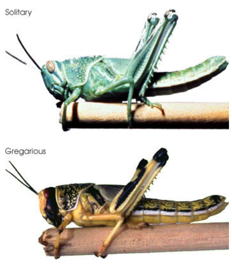
Outline:
- The Observability Problem
- Switched Linear Systems
- Pathwise Observability
- Non-Pathological Sampling
- Mode Observability
- Observer Design



Joint work with Mohamad Babaali

Why Hybrid Observers and Observability?

- A number of naturally occurring as well as engineered systems exhibit mode switches.

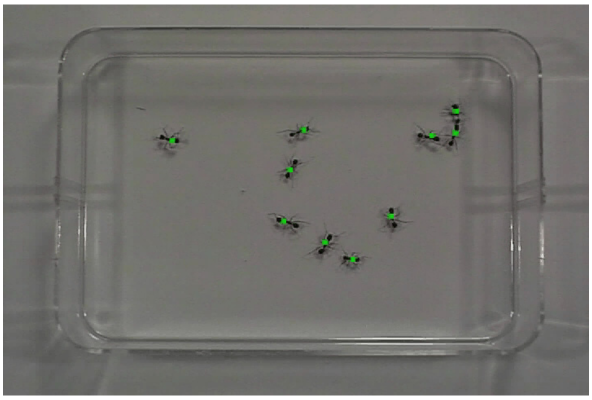


Naturally occurring switches: Can we figure out what mode a system is in, as well as what triggers the transitions?

Switches by Design: Can we estimate the (continuous) state of the system in order to select appropriate modes of operation?

Switches by Necessity: When switches occur as a reaction to the environment, can we simultaneously find the mode as well as the continuous state?

Why Hybrid Observers and Observability?



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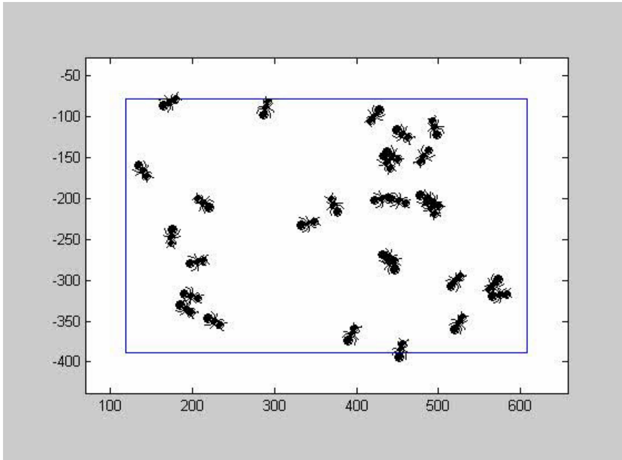

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Kalman Revisited

- In the 50s and 60s, Kalman wrote a series of papers (e.g. [Kalman'63]) in which he considered the problem of finding the initial state of a system, given a sequence of measured outputs

System: $\begin{cases} x_{k+1} = Ax_k \in \Re^n \\ y_k = Cx_k \in \Re^k \end{cases}$ | Given $y_0, \ldots, y_N$ Find x_0

- Kalman's solution:

$$\begin{array}{l} y_0 = Cx_0 \\ y_1 = Cx_1 = CAx_0 \\ y_2 = Cx_2 = CA^2x_0 \\ \vdots \\ y_N = Cx_N = CA^Nx_0 \end{array} \Rightarrow \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^N \end{bmatrix} x_0 \text{ or } \mathbf{Y}_{N+1} = \mathcal{O}_{N+1}x_0$$

Kalman Revisited

- Let $\rho(M) = \text{rank}(M)$
- Assume that we have made only one measurement $\mathbf{Y}_1 = \mathcal{O}_1 x_0$
- The initial state can be recovered if:

$$k = n \text{ and } \rho(\mathcal{O}_1) = n \Rightarrow x_0 = \mathcal{O}_1^{-1}\mathbf{Y}_1$$

$$k > n \text{ and } \rho(\mathcal{O}_1) = n \Rightarrow x_0 = \mathcal{O}_1^{\{1\}}\mathbf{Y}_1$$

$k < n$ no way…

$\{1\}$-inverse (see next page)

- And similarly for ***N*** steps – The initial state can be recovered in ***N*** steps if $\rho(\mathcal{O}_N) = n$
- **Algebraic Characterization** of observability

$$\text{Observability} \Leftrightarrow \exists N \text{ s.t. } \rho(\mathcal{O}_N) = n$$

Some Matrix Inversion Theory

- A $\{1\}$-inverse (from matrix inversion theory) is a generalization of the Moore-Penrose pseudo-inverse.
- $M^{\{1\}}$ is a $\{1\}$-inverse of M if

$$MM^{\{1\}}M = M$$

while the Moore-Penrose (also a $\{1\}$-inverse) satisfies

$$MM^{\dagger}M = M,\ M^{\dagger}MM^{\dagger} = M^{\dagger},\ M^{\dagger}M = (M^{\dagger}M)^T,\ MM^{\dagger} = (MM^{\dagger})^T$$

- If M has full column rank then any $\{1\}$-inverse is also a left-inverse

$$M^{\{1\}}M = I$$

- So, returning to our observability problem, where

$$\mathbf{Y} = \mathcal{O}x_0$$

- Given the measurements, find the initial state?

Some Matrix Inversion Theory

- The system

$$\mathbf{Y} = \mathcal{O}x$$

has a solution if and only if $\mathbf{Y} \in \mathcal{R}(\mathcal{O})$. If such a solution exists, then

$$x = \mathcal{O}^{\{1\}}\mathbf{Y}$$

for any $\{1\}$-inverse.
- Moreover, the solution is unique if and only if $\mathcal{O}$ has full rank.

Kalman Revisited

- Fine, but, the question is how "far" we have to go (i.e. how many measurements are needed) before we either can recover the initial state or give up because we know we will never be able to do so?
- Kalman's clever answer:

$$\exists N \text{ s.t. } \rho(\mathcal{O}_N) = n \Leftrightarrow \rho(\mathcal{O}_n) = n$$

(Thanks to Cayley-Hamilton's theorem)

- This can in fact be thought of as a **Computational Complexity** Characterization of observability
- And, this rather odd view of Kalman's result will turn out to be useful when treating observability of hybrid systems

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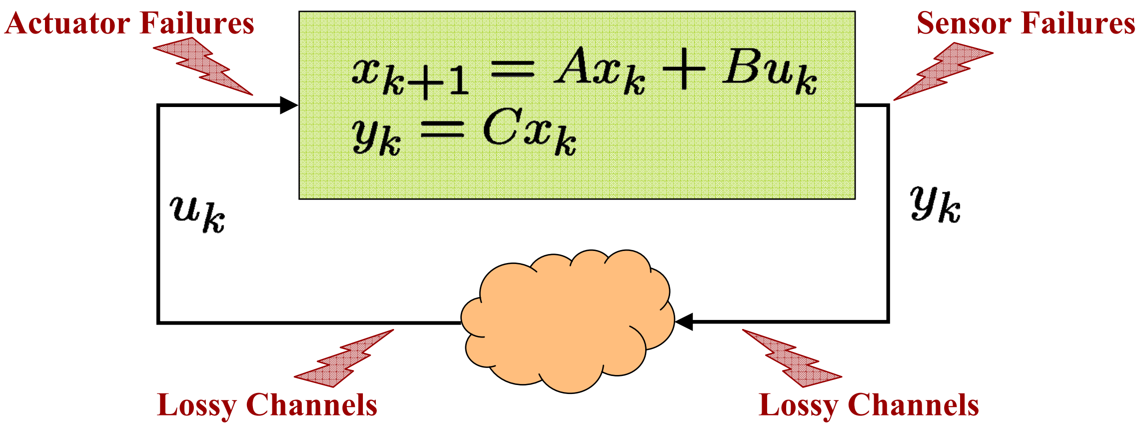
Switched Linear Systems

- As an example, consider the system



- We want to be able to do estimation and control in the presence of such limitations/switching effects

Switched Linear Systems

- Such situations can be modeled as **switched linear systems**.
- Let the ***mode*** of the system at time *k* be

$$\theta_k \in \Theta = \{1, 2, \ldots, s\} \longleftarrow \text{mode set}$$

- We have that

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k$$
$$y_k = C(\theta_k)x_k$$

- *When can we recover the state (and possibly the mode – if unknown) for such systems?*

Some Notation

- Given the mode set Θ (the **alphabet**)
- Elements in Θ can be concatenated together to form **words**
- The set of all finite length words over Θ (including the "empty word" ε) is denoted by $\Theta^\star$
- Relative to the concatenation operation, $\Theta^\star$ is a monoid (a semi-group with identity)

$$\text{closure:} \quad \sigma_1, \sigma_2 \in \Theta^\star \;\Rightarrow\; \sigma_1 \cdot \sigma_2 \in \Theta^\star$$
$$\text{identity:} \quad \varepsilon \cdot \sigma = \sigma \cdot \varepsilon = \sigma, \;\forall \sigma \in \Theta^\star$$

- A **formal language** (e.g. [Hopcroft'00]) over Θ is any subset of words, i.e. $\mathcal{L} \subset \Theta^\star$

- Given a string of N modes $\hat{\theta}_N = \theta_0 \cdot \theta_1 \cdots \theta_{N-1}$ $\quad (\hat{\theta}(i) = \theta_{i-1})$
- We say that $\hat{\theta}_N \in \Theta_N = \{\sigma \in \Theta^\star \mid |\sigma| = N\}$ (language with all words of length N)

length

Switched Linear Systems

- Assume that the evolution of the modes gives rise to the following mode string $\hat{\theta}_N = \theta_0 \cdot \theta_1 \cdots \theta_{N-1}$
- We get

(Assume no ***B*** matrix)

$$
\begin{aligned}
y_0 &= C(\theta_0)x_0 \\
y_1 &= C(\theta_1)x_1 = C(\theta_1)A(\theta_0)x_0 \\
y_2 &= C(\theta_2)x_2 = C(\theta_2)A(\theta_1)x_1 = C(\theta_2)A(\theta_1)A(\theta_0)x_0 \\
&\vdots \\
y_{N-1} &= C(\theta_{N-1})x_{N-1} = C(\theta_{N-1})A(\theta_{N-2})\cdots A(\theta_0)x_0
\end{aligned}
$$

- Let the observability matrix of $\hat{\theta}_N$ be

$$\mathcal{O}(\hat{\theta}_N) = \begin{pmatrix} C(\theta_0) \\ \vdots \\ C(\theta_{N-1})A(\theta_{N-2})\cdots A(\theta_0) \end{pmatrix}$$

Some Key Concepts and Questions

- **Autonomous Systems** (no ***B***-matrix)
  - ***State Observability***: Given that the mode sequence is known/directly observable, find *x*
    - ***Universal observability***: Can we find *x* for all mode sequences (after enough steps)? I.e.

      $$\exists M \text{ s.t. } \rho(\mathcal{O}(\hat{\theta}_M)) = n \ \forall \hat{\theta}_M \in \Theta_M?$$

    - ***Existential observability:*** Does there exist a mode sequence such that we can recover *x*? I.e.

      $$\exists \hat{\theta} \in \Theta^\star \text{ s.t. } \rho(\mathcal{O}(\hat{\theta})) = n?$$

    - ***Hybrid observability:*** Given that the mode sequence is driven by the *x*-trajectory, can we find *x*?

Some Key Concepts and Questions

- **Autonomous Systems** (no ***B***-matrix)
  - ***Mode Observability***:
    - ***Known state, mode-observability***: Given that the continuous state is known/directly accessible, can we find the mode sequence?
    - ***Unknown state, mode-observability:*** Given that we only observe the output of the system, can we recover the mode sequence?
  - ***Joint State-Mode Observability***
- **Non-Autonomous Systems?**
- **Observer Design?**

What's Really in this Presentation?

- We will discuss the problem of universal state-observability with known mode sequences in detail for autonomous systems.
- The main results for the other autonomous cases will be recalled.
- For non-autonomous systems, the lack of a separation principle will be discussed and hardly any crisp results (this is not a completely understood area yet) will be given
- A collection of observers will be designed for switched linear systems with known as well as unknown mode sequences.

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Universal Observability

- Given

$$x_{k+1} = A(\theta_k)x_k$$
$$y_k = C(\theta_k)x_k$$
$$\theta_k \in \Theta = \{1, 2, \ldots, s\}$$

where the mode sequence is *arbitrary*, *known*, and *exogenous*, can we find the initial state?

- **Def:** The set of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ is **pathwise observable** if

$$\exists M \text{ s.t. } \rho(\mathcal{O}(\widehat{\theta}_M)) = n \ \forall \widehat{\theta}_M \in \Theta_M$$

- The smallest such integer is the ***index*** of PWO and is denoted by $\mathcal{N}_{PWO}$

Pathwise Observability

- Some examples…

$$A(1) = A(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ C(1) = (1, 0), \ C(2) = (2, 0)$$

$$\rho(1 \cdot 1) = \rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2, \ \rho(2 \cdot 2) = \rho \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2$$
$$\rho(1 \cdot 2) = \rho \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2, \ \rho(2 \cdot 1) = \rho \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

- This system is PWO with index 2!

Pathwise Observability

- Some examples…

$$A(1)=A(2)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ C(1)=(1,0),\ C(2)=(0,1)$$

$$\rho(1\cdot 1)=\rho\begin{pmatrix}1&0\\0&1\end{pmatrix}=2$$

$$\rho(2\cdot 2)=\rho\begin{pmatrix}0&1\\1&0\end{pmatrix}=2$$

$$\rho(1\cdot 2\cdot 1\cdot 2\cdots)=\rho\begin{pmatrix}1&0\\1&0\\\vdots&\vdots\\1&0\end{pmatrix}=1$$

- This system is not PWO!

Pathwise Observability

- Some examples…

$$A(1)=A(2)=\begin{pmatrix}1&1\\1&0\end{pmatrix},\ C(1)=(1,-1),\ C(2)=(1,0)$$

$$\rho(1\cdot 1)=\rho\begin{pmatrix}1&-1\\1&0\end{pmatrix}=2,\ \rho(2\cdot 2)=\rho\begin{pmatrix}1&0\\1&1\end{pmatrix}=2$$

$$\rho(1\cdot 2)=\rho\begin{pmatrix}1&-1\\1&1\end{pmatrix}=2,\ \rho(2\cdot 1)=\rho\begin{pmatrix}1&0\\1&0\end{pmatrix}=1$$

$$\rho(2\cdot 1\cdot 1)=\rho\begin{pmatrix}1&0\\1&0\\1&1\end{pmatrix}=2,\ \rho(2\cdot 1\cdot 2)=\rho\begin{pmatrix}1&0\\1&0\\1&2\end{pmatrix}=2$$

- This system is PWO with index 3!

A Sufficient Condition

- In general, it is quite hard to test for PWO, so we start with a simpler case:
- Let the system matrix $\boldsymbol{A}$ be constant and assume that somewhere in the mode sequence we have

$$\cdots i \cdots i \cdots\cdots i \cdots$$

after $a-1$ modes

b modes apart

repeated n times

- If this was the case then we could "pull out" the observability submatrix

$$\begin{pmatrix} C(i) \\ C(i)A^b \\ \vdots \\ C(i)(A^b)^{n-1} \end{pmatrix} A^{a-1}$$

A Sufficient Condition

$$\cdots i \cdots i \cdots\cdots i \cdots$$

after $a-1$ modes

b modes apart

repeated n times

$$\begin{pmatrix} C(i) \\ C(i)A^b \\ \vdots \\ C(i)(A^b)^{n-1} \end{pmatrix} A^{a-1}$$

- What's going on here?

$$\mathcal{O} = \begin{pmatrix} C(\theta_0) \\ \vdots \\ C(\theta_{a-1})A^{a-2} \\ C(i)A^{a-1} \\ C(\theta_{a+1})A^{a} \\ \vdots \\ C(i)A^{b+(a-1)} \\ \vdots \\ \vdots \\ C(i)A^{(n-1)b+(a-1)} \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} C(i)A^{a-1} \\ C(i)A^{b+(a-1)} \\ \vdots \\ C(i)A^{(n-1)b+(a-1)} \end{pmatrix}$$

$$\begin{pmatrix} C(i) \\ C(i)A^b \\ \vdots \\ C(i)A^{(n-1)b} \end{pmatrix} A^{a-1} = \mathcal{O}' A^{a-1}$$

Note: If $\boldsymbol{A}$ is invertible then $\rho(\mathcal{O}') = n \Rightarrow \rho(\mathcal{O}) = n$

A Sufficient Condition

$\cdots i \cdots i \cdots\cdots\cdots i \cdots$

after $a-1$ modes

b modes apart

repeated n times

$$\begin{pmatrix} C(i) \\ C(i)A^b \\ \vdots \\ C(i)(A^b)^{n-1} \end{pmatrix} A^{a-1}$$

- The basic idea here is to let observability be driven by the individual modes rather by the way in which they mix.
- The question is, of course, whether such “arithmetic progressions” always take place?
- And, surprisingly enough, this question has been answered in a branch of mathematics called Ramsey Theory, in which the “amount of disorder” that one can squeeze into finite sets.

A Sufficient Condition

- **Theorem** [van der Waerden’27]

  For every positive integers n and s, there exists a minimal constant $\mathcal{W}(n,s)$ such that if $N \geq \mathcal{W}(n,s)$ and $\{1,\ldots,N\} \subset C_1 \cup \cdots \cup C_s$, then some set C_i contains an arithmetic progression of length n.

- **Corollary**

  Let $\hat{\theta}$ be a path assuming values in $\{1,\ldots,s\}$. If $|\hat{\theta}| \geq \mathcal{W}(n,s)$ then there exists an integer $i \in \{1,\ldots,s\}$ and two positive integers a and b such that $\hat{\theta}(a+bk) = i$ for every $k = 0,\ldots,n-1$.

- **Corollary** [Babaali’05]

  If A is invertible and $(A^l, C(i))$ is an observable pair for all $i \in \{1,\ldots,s\}$ and all positive integers $l \leq \mathcal{W}'(n,s)$ then $\{(A,C(1)),\ldots,(A,C(s))\}$ is PWO with an index no larger than $\mathcal{W}(n,s)$.

Non-Pathological Sampling

- **Theorem (Kalman-Bertram Criterion)** [Kalman'63]

Let $\sigma(A)$ denote the spectrum of A. If (A, C) is an observable pair, then whenever the sampling period T satisfies, for all $\{\lambda, \lambda'\} \in \sigma(A) \times \sigma(A)$,

$$\lambda \neq \lambda' + \frac{ik}{T}, \ \forall k \in Z \backslash \{0\},$$

then the discrete time pair (e^{AT}, C) is observable.

- **Corollary** [Babaali'05]

If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then whenever

$$\lambda \neq \lambda' + \frac{ik}{lT}, \ \forall k \in Z \backslash \{0\}, \ \forall l \leq \mathcal{W}'(n, s)$$

the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$ of the discretized system is PWO with an index no larger than $\mathcal{W}(n, s)$.

Non-Pathological Sampling

The moral of the story here is that for sampled systems, as long as the individual pairs are observable for the continuous time system, the sampled system is (almost always) PWO!

Decidability

- Now, let's return to the general case
- **Theorem:** [Babaali'03]

> For all s and n there exist a number $\mathcal{N}(s,n)$ such that for any set of s pairs
> $$\{(A(1),C(1)),\ldots,(A(s),C(s))\}$$
> if the set is pathwise observable then the index of PWO is smaller than or equal to $\mathcal{N}(s,n)$.

- Unfortunately, these uniform bounds are huge (towers – exponentials of exponentials of exponentials…).

$$\mathcal{N}(2,3) = 135, \;\; \mathcal{N}(4,4) \approx 3.3 \cdot 10^{619}$$

- However, the existence of these bounds do establish that **PWO is in fact decidable!**

So Far…

- PWO is decidable!
- The result trivially carries over to pathwise controllability: PWO of the set of dual pairs.
- Exact determination of (or lower bounds on) the index of PWO is still an open issue.
- **PWO only gives the initial state if the modes are known.**

Some Key Concepts and Questions (Again)

- **Autonomous Systems** (no ***B***-matrix)
  - ***State Observability***: Given that the mode sequence is known/directly observable, find x
    - ***Universal observability***: Can we find x for all mode sequences (after enough steps)? I.e.

      $$\exists M \text{ s.t. } \rho(\mathcal{O}(\hat{\theta}_M)) = n \ \forall \hat{\theta}_M \in \Theta_M?$$

      Decidable!
    - ***Existential observability:*** Does there exist a mode sequence such that we can recover x? I.e.

      $$\exists \hat{\theta} \in \Theta^\star \text{ s.t. } \rho(\mathcal{O}(\hat{\theta})) = n?$$
    - ***Hybrid observability:*** Given that the mode sequence is driven by the x-trajectory, can we find x?

The Existential Problem

$$\exists \hat{\theta} \in \Theta^\star \text{ s.t. } \rho(\mathcal{O}(\hat{\theta})) = n?$$

- If the Universal Problem can be thought of as a robustness against any type of switching property, the Existential Problem involves *actively designing a switching sequence that renders the system observable*.
- And, luckily for us, this turns out to be significantly easier to "solve" in the sense that

> **Thm:** [Egerstedt'05] If a mode sequence $\hat{\theta} \in \Theta^\star$ exists such that $\rho(\mathcal{O}(\hat{\theta})) = n$ for the system
> $$x_{k+1} = Ax_k, \ y = C(\theta_k)x_k$$
> then such a sequence exists with length $\leq n^2$

- This theorem is also constructive in that it gives us the switching sequence. However, if the ***A*** matrix is allowed to switch as well, this is still an open problem!

Some Key Concepts and Questions (Again)

- **Autonomous Systems** (no ***B***-matrix)
  - ***State Observability***: Given that the mode sequence is known/directly observable, find *x*
    - ***Universal observability***: Can we find *x* for all mode sequences (after enough steps)? I.e.

$$\exists M \text{ s.t. } \rho(\mathcal{O}(\hat{\theta}_M)) = n \ \forall \hat{\theta}_M \in \Theta_M?$$

Decidable!

- ***Existential observability:*** Does there exist a mode sequence such that we can recover *x*? I.e.

$$\exists \hat{\theta} \in \Theta^\star \text{ s.t. } \rho(\mathcal{O}(\hat{\theta})) = n?$$

Decidable!

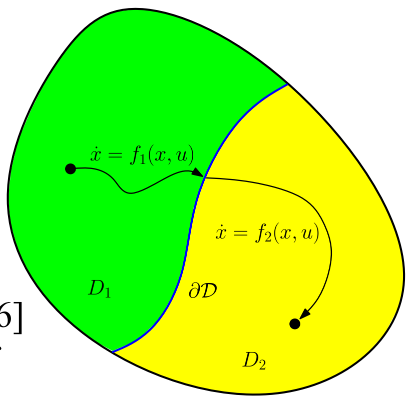
(sometimes?)

- ***Hybrid observability:*** Given that the mode sequence is driven by the *x*-trajectory, can we find *x*?

Hybrid Observability

- Let the switches between modes occur when a given switching surface is reached, e.g.

$$x_{k+1} = A(\theta_k)x_k$$
$$y_k = C(\theta_k)x_k$$
$$\theta_k = \begin{cases} 1 & \text{if } d^T x_k < 0 \\ 2 & \text{if } d^T x_k \geq 0 \end{cases}$$



- Unfortunately, Eduardo Sontag [Sontag'96] showed that the problem of establishing if this system is observable is (in general) undecidable!

Some Key Concepts and Questions (Again)

- **Autonomous Systems** (no ***B***-matrix)
  - *State Observability*: Given that the mode sequence is known/directly observable, find x

Decidable!

- *Universal observability*: Can we find x for all mode sequences (after enough steps)? I.e.

$$\exists M \text{ s.t. } \rho(\mathcal{O}(\hat{\theta}_M)) = n \ \forall \hat{\theta}_M \in \Theta_M?$$

Decidable! (sometimes?)

- ***Existential observability:*** Does there exist a mode sequence such that we can recover x? I.e.

$$\exists \hat{\theta} \in \Theta^{\star} \text{ s.t. } \rho(\mathcal{O}(\hat{\theta})) = n?$$

Undecidable!

- ***Hybrid observability:*** Given that the mode sequence is driven by the x-trajectory, can we find x?

And Who are (some of the) Key Players Here?

- **Decidability of PWO**: [Gurvits'02, Babaali'03].
- **PWO for slowly varying discrete time systems**: [Vidal'02, Balluchi'03]
- **PWO for continuous time systems:** [De Santis'03, Vidal'03]
- **PWO for sampled systems:** [Babaali'05]
- **Existential observability:** [Egerstedt'05, Gurvits'02, Sun'02, Stanford'80]
- **Hybrid observability:** [Sontag'96, Bemporad'00, Collins'04, Ferrari-Trecate'02]

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Some Key Concepts and Questions (again)

- **Autonomous Systems** (no ***B***-matrix)
  - *Mode Observability*:
    - ***Known state, mode-observability***: Given that the continuous state is known/directly accessible, can we find the mode sequence?
    - ***Unknown state, mode-observability:*** Given that we only observe the output of the system, can we recover the mode sequence?
  - *Joint State-Mode Observability*
- Turns out that we can answer the second question just as easily as the first question so, why not just do that right away?
- The joint question will be answered as part of the observer design section.

Unknown Modes (Autonomous)

$$x_{k+1} = A(\theta_k)x_k, \; y_k = C(\theta_k)x_k$$
$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \mathcal{O}(\widehat{\theta})x_1$$

- Mode Observability at ***N***: Want to determine the first ***N*** modes from the first ***N+N'*** observations, i.e. determine

$$\widehat{\theta}_N \text{ from } \mathcal{O}(\widehat{\theta}_N \cdot \widehat{\theta}_{N'})x_1$$

- Theorem: [Babaali'04] MO at ***N*** is generically (e.g. the initial state is non-zero) decidable…
- Corollary: State Observability at ***N*** is also generically decidable…

Mode Observability

- That's really all that will be covered about this at this point. (More to follow in the section on Observers…)
- Some useful references:
  - **Decidability of MO:** [Gurvits'02, Babaali'04]
  - **MO for continuous time systems:** [De Santis'03, Vidal'03]

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Non-Autonomous Systems

- If there is in fact a ***B***-term as well, we get

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k$$
$$y_k = C(\theta_k)x_k$$

- Since there is no separation principle any more, e.g.

$$\hat{x}_{k+1} = A(\theta_k)\hat{x}_k + B(\theta_k)u_k + L(\theta_k)(y_k - C(\theta_k)\hat{x}_k)$$

?

we can't separate the control problem from the estimation problem.

- Two interesting (and largely ***open***) questions:
  - Is the system observable for (almost) any inputs?
  - Can we select inputs in such a way that we achieve observability?
- This part is of the presentation is even shorter still and rather than giving crisp results, the few existing, relevant references are: [Babaali'04, Babaali'05b, De Santis'03]

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Observers

- So far we have really only looked at the problem of deciding if we can in fact figure out the state of a hybrid system.
- But, given that it is possible, how should one do it?
- A number of clever observer structures have been proposed and here we will focus our attention on two particular choices:
  - **The Direct Algebraic Approach** (we don't bother trying to recover the mode but go directly for the continuous state)
  - **The Two-Stage Observer** (first we try and find the mode and then, based on this, we recover the state)
- A number of other solutions have been proposed, but these two serve well as illustrations to the two main design philosophies

Direct Observers

$$x_{k+1} = Ax_k,\ \ y_k = C(\theta_k)x_k,\ \ y_k \in \mathbf{R}$$

- **Problem:** Design a convergent observer for this system under unknown and arbitrary mode sequences:

$$\hat{x}_{k+1} = F(\hat{x}_k, y_k, \ldots, y_{k+N-1})$$
$$\|\hat{x}_k - x_k\| \rightarrow 0 \text{ if } \|\hat{x}_0 - x_0\| \leq \delta$$

- **What's hard about this?:**
  - We don't know the mode sequence. And, as such, we can not choose the "correct" Luenberger observer by mimicking/observing the switching mechanism.
- **A (smart) trick:**

$$y_k = C(\theta_k)x_k \text{ for some } \theta_k \Leftrightarrow \prod_{i=1}^{s}(y_k - C(i)x_k) = 0$$

Direct Observers

$$x_{k+1} = Ax_k$$
$$\prod_{i=1}^{s}(y_k - C(i)x_k) = 0$$

- **Deterministic model:** Circumventing the need to figure out the mode in effect.
- **Nonlinear model:** Convergence issues (multiple solutions)

- **Choice:** Newton Observers [Moraal'95]

$$G_k(x) = \begin{pmatrix} \prod_{i=1}^{s}(y_k - C(i)x) \\ \vdots \\ \prod_{i=1}^{s}(y_{k+N-1} - C(i)A^{N-1}x) \end{pmatrix},\ N \geq n$$

- **Idea:** Run a single Newton iteration at each time

Newton Observers - Basics

- Assume that we have a nonlinear system
$$x_{k+1} = f(x_k),\ \ y_k = h(x_k)$$
- At each time (given the output) we would like to find the solution to
$$\gamma(x) = y - h(x) = 0$$
- Or, "equivalently", we'd like to minimize
$$\Gamma(x) = \|\gamma(x)\|^2$$
- Newton's iterative method tells us that we should pick
$$x_{\ell+1} = x_\ell - \left(\frac{\partial^2\Gamma}{\partial x^2}(x_\ell)\right)^\dagger \frac{\partial\Gamma}{\partial x}(x_\ell)$$
or "equivalently"
$$x_{\ell+1} = x_\ell - \frac{\partial\gamma(x_\ell)}{\partial x}\gamma(x_\ell)$$

Direct Observers

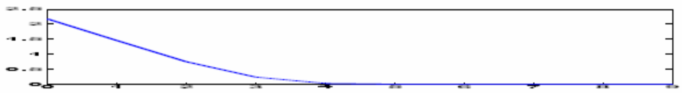
- **DAA-Newton Observer:**
$$\hat{x}_k^- = A\hat{x}_{k-1} \quad \longleftarrow \text{predictor}$$
$$\hat{x}_k = \hat{x}_k^- - \left(G_k'(\hat{x}_k^-)\right)^\dagger G_k(\hat{x}_k^-) \longleftarrow \text{corrector}$$
- Main issues:
  1. x_k unique local solution of $G_k(x_k) = 0$?
  2. $G_k'(x_k)$ full rank?
- **Theorem:** [Babaali'04b]

  Technical assumption ($\sim$ PWO) + $x_0 \neq 0$ + A invertible $\Rightarrow$ DAA-Netwon observer is locally, exponentially convergent

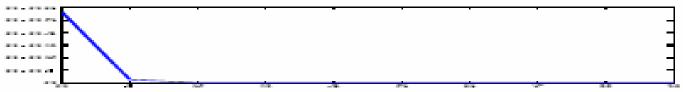
Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; C(1) = (1,0), \; C(2) = (2,3)$$

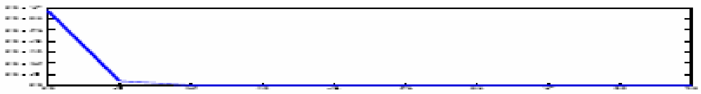
- Theorem requirements satisfied with $N = 3$
  - $\theta = 1 \cdot 1 \cdot 1 \cdot 1 \cdots$



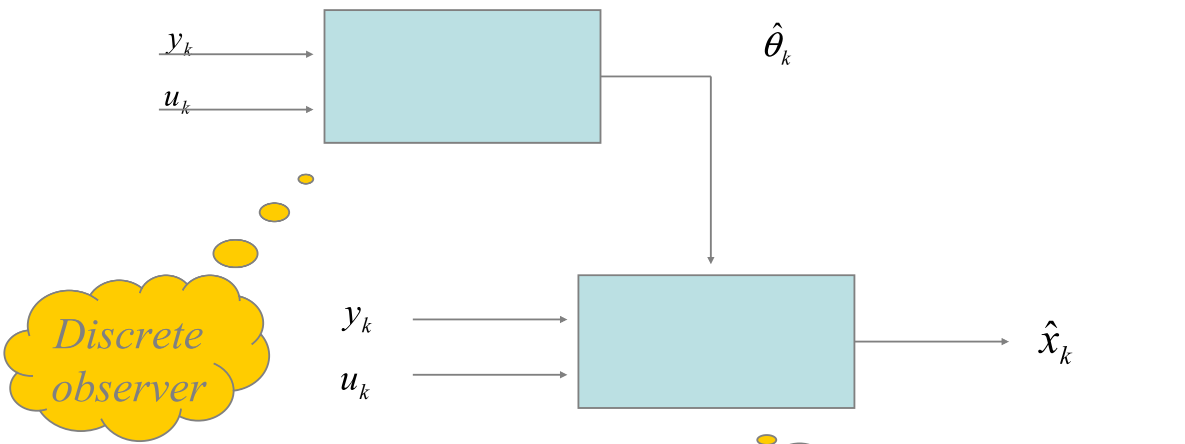
  - $\theta = 2 \cdot 2 \cdot 2 \cdot 2 \cdots$



  - $\theta = 1 \cdot 2 \cdot 1 \cdot 2 \cdots$



Two-Stage Observers



- Note that in the previous construction, the mode (although unknown) is never explicitly computed.
- An alternative (and somewhat popular) approach is to fist find the mode, and then the continuous state,

Two-Stage Observers

- The main idea is to, at time k, use the previous N measurements to figure out the mode at time k. I.e. use the finite horizon:

$$\text{For } \hat{\theta}_k, \text{ use only } u_i, y_i, \ i = k - N + 1, \ldots, k$$

- Once this mode is known, one can use a Luenberger-type observer for the continuous state

$$\hat{x}_{k+1} = A(\hat{\theta}_k)\hat{x}_k + B(\hat{\theta}_k)u_k + L(\hat{\theta}_k)(y_k - C(\hat{\theta}_k)\hat{x}_k)$$

- Note, switched ***stability issues*** for hybrid systems are still important for the estimation error, even if the mode is estimated correctly. (This is a whole other subject…)

- As such we will focus our attention on the discrete observer.

Discrete Observers

- Let

$$\theta_k^N = \theta_{k-N+1}, \ldots, \theta_k$$

$$\mathbf{Y}_k^N = \begin{pmatrix} y_{k-N+1} \\ \vdots \\ y_k \end{pmatrix}$$

$$\mathcal{O}(\theta_k^N) = \begin{pmatrix} C(\theta_{k-N+1}) \\ \vdots \\ C(\theta_k)A(\theta_{k-1})\cdots A(\theta_{k-N+1}) \end{pmatrix}$$

- Note that $\mathbf{Y}_k^N = \mathcal{O}(\theta_k^N)x_{k-N+1}$
- **"Algorithm"** [Ragot'03]

range

$$\begin{cases} \text{Assume:} & \hat{\theta}_i = \theta_i, \ i \leq k-1 \\ \text{Set:} & \hat{\Theta}_k = \{\theta \in \Theta \mid \mathbf{Y}_k^N \in \mathcal{R}(\mathcal{O}(\hat{\theta}_{k-1}^{N-1} \cdot \theta))\} \\ \text{If:} & |\hat{\Theta}_k| = 1 \text{ set } \hat{\theta}_k = \arg\hat{\Theta}_k \end{cases}$$

Discrete Observers

$$\begin{cases} \text{Assume:} & \hat{\theta}_i = \theta_i,\ i \leq k-1 \\ \text{Set:} & \hat{\Theta}_k = \{\theta \in \Theta \mid \mathbf{Y}_k^N \in \mathcal{R}(\mathcal{O}(\hat{\theta}_{k-1}^{N-1} \cdot \theta))\} \\ \text{If:} & |\hat{\Theta}_k| = 1 \text{ set } \hat{\theta}_k = \arg\hat{\Theta}_k \end{cases}$$

- When does this algorithm return a singleton, i.e. when is it convergent?
- **Def:** A path $\tilde{\theta}$ is ***discernible*** from another path $\tilde{\theta}'$ of the same length if

$$\rho([\mathcal{O}(\tilde{\theta})\mathcal{O}(\tilde{\theta}')]) > \rho(\mathcal{O}(\tilde{\theta}'))$$

- **Def:** A mode $\theta \in \Theta$ is ***backward-discernible*** from $\theta' \in \Theta$ if there exists a horizon ***N*** such that for any path $\lambda \in \Theta^\star$ of length ***N***, $\lambda \cdot \theta$ is discernable from $\lambda \cdot \theta'$

- We say that ***N*** is the "index" of backward-discernibility.

Discrete Observers

- **Thm:** [Babaali'05c] The mode detector is well-posed (and convergent) for
  - all mode sequences
  - all control input signals
  - almost all initial states

  if and only if every mode is backward discernible from every other mode with index smaller that the detection horizon.

- And, as a lucky "coincidence", this is in fact checkable using a version of the previous large numbers (exponential of exponentials…)

Example

$C(1) = (1, 0) \quad C(2) = (2, 0)$

$A(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \quad A(2) = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$

Since the error dynamics are the same in both modes, the overall system constitutes a convergent, continuous observer. (Common Lyapunov function)

Mode 1:

Continuous Observer:
$\hat{x}_{k+1} = A(1)\hat{x}_k + L(1)(y_k - C(1)x_k), \; L(1) = (0.5, 0)^T$

Error Dynamics:
$e_k = x_k - \hat{x}_k, \; e_{k+1} = (A(1) - L(1)C(1))e_k = 0.5Ie_k$

Mode 2:

Continuous Observer:
$\hat{x}_{k+1} = A(2)\hat{x}_k + L(2)(y_k - C(2)x_k), \; L(2) = (1.5, 0)^T$

Error Dynamics:
$e_k = x_k - \hat{x}_k, \; e_{k+1} = (A(2) - L(2)C(2))e_k = 0.5Ie_k$

Example

$\mathbf{Y}_0^0 = 1$

$$\begin{cases} \mathcal{O}(1) = C(1) = (1,0), \; \mathcal{R}(\mathcal{O}(1)) = \Re \\ \mathcal{O}(2) = C(2) = (2,0), \; \mathcal{R}(\mathcal{O}(2)) = \Re \end{cases} \Rightarrow \hat{\theta}_0 = 1 \text{ or } \hat{\theta}_0 = 2$$

$\mathbf{Y}_1^0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{cases} \mathcal{O}(1 \cdot 1) = \begin{pmatrix} C(1) \\ C(1)A(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \; \mathcal{R}(\mathcal{O}(1 \cdot 1)) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \\ \mathcal{O}(1 \cdot 2) = \begin{pmatrix} C(1) \\ C(2)A(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \; \mathcal{R}(\mathcal{O}(1 \cdot 2)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \\ \mathcal{O}(2 \cdot 1) = \begin{pmatrix} C(2) \\ C(1)A(2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \; \mathcal{R}(\mathcal{O}(2 \cdot 1)) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \\ \mathcal{O}(2 \cdot 2) = \begin{pmatrix} C(2) \\ C(2)A(2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}, \; \mathcal{R}(\mathcal{O}(2 \cdot 2)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \end{cases}$$

$$\Rightarrow \hat{\theta}_0 \cdot \hat{\theta}_1 = 1 \cdot 2 \text{ or } \hat{\theta}_0 \cdot \hat{\theta}_1 = 2 \cdot 2$$

Example

$$\mathbf{Y}_2^0 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{cases} \mathcal{O}(1\cdot2\cdot1) = \begin{pmatrix} C(1) \\ C(2)A(1) \\ C(1)A(2)A(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 2 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(1\cdot2\cdot1)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right\} \\ \mathcal{O}(1\cdot2\cdot2) = \begin{pmatrix} C(1) \\ C(2)A(1) \\ C(2)A(2)A(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(1\cdot2\cdot2)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right\} \\ \mathcal{O}(2\cdot2\cdot1) = \begin{pmatrix} C(2) \\ C(2)A(2) \\ C(1)A(2)A(2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 4 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(2\cdot2\cdot1)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right\} \\ \mathcal{O}(2\cdot2\cdot2) = \begin{pmatrix} C(2) \\ C(2)A(2) \\ C(2)A(2)A(2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 8 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(2\cdot2\cdot2)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right\} \end{cases}$$

$$\Rightarrow \hat{\theta}_0\cdot\hat{\theta}_1\cdot\hat{\theta}_2 = 1\cdot2\cdot2 \text{ or } \hat{\theta}_0\cdot\hat{\theta}_1\cdot\hat{\theta}_2 = 2\cdot2\cdot2$$

Example

$$\mathbf{Y}_3^0 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 16 \end{pmatrix} \begin{cases} \mathcal{O}(1\cdot2\cdot2\cdot1) = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \\ 8 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(1\cdot2\cdot2\cdot1)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}\right\} \\ \mathcal{O}(1\cdot2\cdot2\cdot2) = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \\ 16 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(1\cdot2\cdot2\cdot2)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \\ 16 \end{pmatrix}\right\} \\ \mathcal{O}(2\cdot2\cdot2\cdot1) = \begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 8 & 0 \\ 8 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(1\cdot2\cdot2\cdot2\cdot1)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}\right\} \\ \mathcal{O}(2\cdot2\cdot2\cdot2) = \begin{pmatrix} 2 & 0 \\ 4 & 0 \\ 8 & 0 \\ 16 & 0 \end{pmatrix}, \ \mathcal{R}(\mathcal{O}(2\cdot2\cdot2\cdot2)) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}\right\} \end{cases}$$

$$\Rightarrow \hat{\theta}_0\cdot\hat{\theta}_1\cdot\hat{\theta}_2\cdot\hat{\theta}_3 = 1\cdot2\cdot2\cdot2$$

Example

- In [Babaali'05c], this system was in fact shown to be backwards discernable with horizon 4.
- As such we know that after 4 steps, the discrete observer is well-posed, i.e. it returns the correct mode!
- Once the correct mode has been recovered, the switched Luenberger observers can be directly applied.

Observers: What Other Results Are There?

- So now we have two observers:
  - A direct observer that does not involve the computation of the discrete mode
  - A two-stage observer that first computes the discrete mode, and then uses a standard observer-structure for the continuous part
- Many other results on observer design for hybrid systems have been developed:
- **Moving horizon observers for piecewise affine hybrid systems:** [Ferrari-Trecate'02, Juloski'03]
- **Switching observers:** [Alessandri'01]
- **Two-stage observers:** [Balluchi'02, Ragot'03]
- **One-stage observer:** [Babaali'04]
- **Observers for stochastic hybrid systems**: Not covered here – But this is an active (and somewhat disjoint) area of research as well

Conclusions

- Interesting things happens with observability (and by duality – controllability) properties of switched systems through the interactions between the different modes
- For non-autonomous systems:
  - Universal Observability is decidable
  - Existential observability is decidable (and constructive)
  - Hybrid observability is undecidable
- Joint state-mode obervability is generically decidable which allows us to design two-stage observers
- For the autonomous case, much work remains to be done
  - Universal properties?
  - "Active sensing"?
- A number of observers have been proposed that either work directly on the continuous state, or has a two-stage character.

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