Stability and stabilization of hybrid systems

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Goals and class structure

**Goal:** After these lectures, you should
- Know some basic theory for stability and stabilization of hybrid systems
- Be familiar with the computational methods for piecewise linear systems
- Understand how the tools can be applied to (relatively) practical systems

Three lectures:
1. Stability theory
2. Computational tools for piecewise linear systems
3. Applications
Part I – Stability theory

Outline:
• A hybrid systems model and stability concepts
• Lyapunov theory for smooth systems
• Lyapunov theory for stability and stabilization of hybrid systems

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A hybrid systems model

We consider hybrid systems on the form
\[ \dot{x}(t) = f(x(t), i(t)) \]
\[ i(t) \downarrow = \nu(x(t), i(t)) \]
where
- \( x(t) \in \mathbb{R}^n \) is the continuous state vector
- \( i(t) \in \{1, 2, \ldots, M\} \) is the discrete state

The discrete state indexes vector fields \( f(x, i) = f_i(x) \) while \( \nu(x, i) \) is the transition function describing the evolution of the discrete state.

Unless stated otherwise, we will assume that \( i(t) \) is piecewise continuous (i.e., that there is only a finite number of mode changes per unit time)

For now, disregard issues with sliding modes, zeno, ... (see refs for details)
Example: a switched linear system

\[ \dot{x}(t) = A_{i(t)}x(t) \]
\[ i(t^+) = \begin{cases} 
2 & \text{if } i(t) = 1 \text{ and } x_2 = -10x_1 \\
1 & \text{if } i(t) = 2 \text{ and } x_2 = 2x_1 
\end{cases} \]

(numerical values for matrices \( A_i \) are given in notes for Lecture 2)

Stability concepts

**Focus:** stability of equilibrium point (in continuous state-space) \( x = 0 \)

**Global asymptotic stability** (GAS): ensure that

\[ \lim_{t \to \infty} x(t) = 0 \quad \text{for all initial states } (x(0), i(0)) \]

**Global uniform asymptotic stability** (GUAS): ensure that

\[ \lim_{t \to \infty} x(t) = 0 \quad \text{for all initial states } (x(0), i(0)) \]

and for all piecewise continuous \( i(t) \)

(i.e., uniformly in \( i(t) \) )
Three fundamental problems

**Problem P1:** Under what conditions is
\[ \dot{x}(t) = f(x(t), i(t)) \]
GAS for all (piecewise continuous) switching signals \( i(t) \)?

**Problem P2:** Given vector fields \( f(x, i) = f_i(x) \), design strategy \( \nu(x, i) \):
\[ \dot{x}(t) = f(x(t), i(t)) \]
\[ i(t^+) = \nu(x(t), i(t)) \]
is globally asymptotically stable.

**Problem P3:** determine if a given switched system
\[ \dot{x}(t) = f(x(t), i(t)) \]
\[ i(t^+) = \nu(x(t), i(t)) \]
is globally asymptotically stable.

Part I – Stability theory

**Outline:**
- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

**Aim:** establishing common grounds by reviewing fundamentals.
Lyapunov theory for smooth systems

Theorem. Let \( x = 0 \) be an equilibrium point of \( \dot{x} = f(x) \), and let \( V : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function such that

(i) \( V(x) \to \infty \) as \( \|x\| \to \infty \) (radially unbounded)
(ii) \( V(0) = 0 \) and \( V(x) > 0 \) if \( x \neq 0 \) (positive definite)
(iii) \( \dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \) for all \( x \neq 0 \) (decreasing)

then \( x = 0 \) is globally asymptotically stable.

**Interpretation:** Lyapunov function is abstract measure of system energy, system energy should decrease along all trajectories.

Converse theorem

Under appropriate technical conditions (mainly smoothness of vector fields)

Theorem. If \( x = 0 \) is a GAS equilibrium of \( \dot{x} = f(x) \), then there exists a radially unbounded Lyapunov function \( V(x) \)

**Consequence:** worthwhile to search for Lyapunov functions

**Remaining challenge:** how to perform Lyapunov function search?
Stability of linear systems

Theorem. The following statements are equivalent:

(i) The linear system $\dot{x} = Ax$ is asymptotically stable

(ii) There is a quadratic Lyapunov function

$$V(x) = x^T P x$$

for some positive definite matrix $P > 0$ such that

$$A^T P + PA < 0$$

Moreover, for every asymptotically stable $A$ and for any $Q > 0$ there is a $P > 0$ such that the following Lyapunov equality holds

$$A^T P + PA = -Q$$

Partial proof

(ii) $\Rightarrow$ (i): Assume that there is $P > 0$ such that $A^T P + PA < 0$. Then there exists an $\epsilon > 0$ such that

$$A^T P + PA + \epsilon P < 0$$

Letting $V(x) = x^T P x$, then for all $t \in \mathbb{R}$

$$\frac{d}{dt} V(x(t)) + \epsilon V(x(t)) = x^T(t)(A^T P + PA)x(t) + \epsilon x^T(t)Px(t)$$

$$= x^T(t)(A^T P + PA + \epsilon P)x(t) \leq 0$$

After integration, this yields for all $t \leq t_0$,

$$x^T(t)Px(t) \leq x^T(t_0)Px(t_0)e^{-\epsilon t}$$

Now use that $\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2$ to infer

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{-\epsilon t}$$
Stability of discrete-time systems

Theorem. Let $x = 0$ be an equilibrium point of $x(t_{k+1}) = f(x(t_k))$, and let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function s.t.

(i) $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$
(ii) $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$
(iii) $\Delta V(x) = V(f(x(t_k))) - V(x(t_k)) < 0$ for all $x \neq 0$

then $x = 0$ is globally asymptotically stable.

Interpretation: energy should decrease at each sampling instant (event)

Performance analysis

Lyapunov techniques also useful for estimating system performance.

Theorem. If there exists a radially unbounded, positive definite storage function $V(x)$ satisfying

$$\frac{\partial V(x)}{\partial x} f(x, w) \leq \gamma^2 \|w\|^2 - \|y\|^2 \quad \forall x, w$$

then the smooth nonlinear system

$$\dot{x}(t) = f(x(t), w(t))$$
$$y(t) = g(x(t))$$

has $L_2$-gain less than $\gamma$ (i.e., $\int_0^t \|y(s)\|^2 \, ds \leq \gamma^2 \int_0^t \|w(s)\|^2 \, ds \quad \forall t$)
Part I – Stability theory

Outline:
• A hybrid systems model and stability concepts
• Lyapunov theory for smooth systems
• Lyapunov theory for stability and stabilization of hybrid systems

Content:
– Guaranteeing stability independent of switching strategy
– Design a stabilizing switching strategy
– Prove stability for a given switching strategy

Switching between stable systems

Q: does switching between stable dynamics always create stable motions?
A: no, not necessarily.

\[ \dot{x} = \Lambda_{i(x)} x \text{ for } x \in X_i \text{ with } \Lambda_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 1 & 10 \\ 0.1 & -1 \end{pmatrix} \]

Subsystems are stable and share the same eigenvalues, but stability depends on switching!
P1: Stability for arbitrary switching signals

**Problem:** when is the switched system

\[ \dot{x}(t) = f(x(t), i(t)) = f_i(x(t)) \]

GAS for all (piecewise continuous) switching signals \( i(t) \)?

**Claim:** only if each subsystem

\[ \dot{x}(t) = f_i(x(t)) \]

admits a radially unbounded Lyapunov function.

(can you explain why?)

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The common Lyapunov function approach

In fact, if the submodels are smooth, the following results hold.

**Theorem.** *If all submodels share a common positive definite radially unbounded Lyapunov function, then the switched system is GUAS.*

**Theorem.** *If the switched system is GUAS, then all submodels share a positive definite radially unbounded common Lyapunov function.*

Hence, common Lyapunov functions necessary and sufficient.
Switched linear systems

For switched linear systems
\[ \dot{x}(t) = A_i(t)x(t) \]
it is natural to look for a common quadratic Lyapunov function
\[ V(x) = x^T P x \quad \text{with } P > 0 \]

\( V(x) \) is a common Lyapunov function if
\[ \dot{V}(x) = x^T (A_i^T P + PA_i) x < 0 \quad \text{for all } i = 1, 2, \ldots, M \]

Such a Lyapunov function can be found by solving linear matrix inequalities
\[ P > 0 \quad A_i^T P + PA_i < 0 \quad \text{for all } i = 1, 2, \ldots, M \]

(systems that admit quadratic \( V(x) \) are called \textit{quadratically stable})

Infeasibility test

It is also possible to prove that there is no common quadratic Lyapunov fcn:

\[ \sum_{i=1}^{M} R_i A_i^T + A_i R_i < 0 \]

\textit{Theorem. If there exist positive definite matrices } R_i > 0 \textit{ such that}

\[ A_i^T P + PA_i < 0 \quad \forall i \in \{1, \ldots, M\} \]
Example

**Question:** Does GUAS of switched linear system imply existence of a common quadratic Lyapunov function?

**Answer:** No, the system given by

\[
A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}
\]

is GUAS, but does not admit any common quadratic Lyapunov function since

\[
R_1 = \begin{pmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{pmatrix}
\]

satisfy the infeasibility condition.

(there is, however, a common *piecewise quadratic* Lyapunov function)

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Example

Sample trajectories of switched system (under two different switching strategies)

Even if solutions are very different, all motions are asymptotically stable.
P2: Stabilization

**Problem:** given matrices $A_i$, find switching rule $\nu(x,i)$ such that

$$\dot{x}(t) = A_{i(t)}x(t)$$

$$i(t^+) = \nu(x(t), i(t))$$

is asymptotically stable.

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Stabilization of switched linear systems

**Theorem.** If there exist $\alpha_i > 0$ with $\sum_i \alpha_i = 1$ such that

$$\dot{x}(t) = \sum_i \alpha_i A_i x(t) := A_{eq} x(t)$$

is globally asymptotically stable, then there exists a switching strategy that makes the switched system globally asymptotically stable.

**Note:** if only two subsystems, then condition is also necessary.
Stabilizing switching rules (I)

State-dependent switching strategy designed from Lyapunov function for $A_{eq}$

Solve Lyapunov equality $A_{eq}^T P + PA_{eq} = -Q$. It follows that

$$\sum_i \alpha_i x^T (A_i^T P + PA_i)x = x^T (A_{eq}^T P + PA_{eq})x = -x^T Q x < 0$$

Thus, for each $x$, at least one mode satisfies $x^T (A_i^T P + PA_i)x(t) < 0$

This implies, in turn, that the switching rule

$$\nu(x) = \arg \min x^T (A_i^T P + PA_i)x$$

is well-defined for all $x$ and that it generates globally asymptotically stable motions.

Stabilizing switching rules (II)

Alternative switching strategy: activate mode $i$ fraction $\alpha_i$ of the time, e.g.,

$$i(t^+) = \begin{cases} 1 & \text{if } 0 \leq t < \alpha_1 T \\ 2 & \text{if } \alpha_1 T \leq t < (\alpha_1 + \alpha_2)T \\ \vdots \\ N & \text{if } \sum_{i=1}^{N-1} \alpha_i T \leq t < T \end{cases}$$

(strategy repeats after duty cycle of $T$ seconds). “Average dynamics” is

$$\dot{x} = A_{eq} x$$

and for sufficiently small $T$ the spectral radius of

$$\exp(A_1 \alpha_1 T) \exp(A_2 \alpha_2 T) \cdots \exp(A_N \alpha_N T)$$

is less than one (i.e., state at beginning of each duty cycle will tend to zero)
Example

Consider the two subsystems given by

\[
A_1 = \begin{pmatrix} -0.5 & 1 \\ 100 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & -100 \\ -0.5 & -1 \end{pmatrix}
\]

Both subsystems are unstable, but the matrix \( A_{eq} = 0.5A_1 + 0.5A_2 \) is stable.

**State-dependent switching:** set \( Q = I \), solve Lyapunov equation to find

\[
P = \begin{pmatrix} 0.5700 & 0.0015 \\ 0.0015 & 0.5728 \end{pmatrix}
\]

**Time-dependent switching:** choose duty cycle \( T \) such that spectral radius of

\[
\exp(A_1T/2) \exp(A_2T/2)
\]

is less than one. Alternate between modes each \( T/2 \) seconds.

Example cont’d

**Time-driven switching**

**State-dependent switching**
P3: Stability for a given switching strategy

**Problem:** how can we verify that the switched system

\[ \dot{x}(t) = f(x(t), i(t)) \]
\[ i(t^+) = \nu(x(t), i(t)) \]

is globally asymptotically stable?

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Stability for given switching strategy

For simplicity, consider a system with two modes, and assume that

\[ \dot{x}(t) = f_i(x(t)) \quad i = 1, 2 \]

are globally asymptotically stable with Lyapunov functions \( V_i \)

Even if there is no common Lyapunov function, stability follows if

\[ V_{i(t_{k-1})}(x(t_k)) = V_{i(t_k)}(x(t_k)) \quad \forall k = 1, 2, \ldots \]

where \( t_k \) denote the switching times.

**Reason:** \( V_i \) is continuous Lyapunov function for the switched system.
Multiple Lyapunov function approach

**Theorem.** Consider the switched system where all submodels \( \dot{x} = f_i(x) \) are globally asymptotically stable with Lyapunov functions \( V_i \).

Suppose that for each pair of switching times \((t_k, t_l)\), \( k < l \) with \( i(t_k) = i(t_l) = i \) and \( i(t_m) \neq i \) for \( t_k < t_m < t_l \), we have

\[
V_i(x(t_l)) \leq V_i(x(t_k)) - \rho(|x(t_k)|)
\]

then the switched system is globally asymptotically stable.

Weaker versions exist:

- No need to require that submodels are stable, sufficient to require that all submodels admit *Lyapunov-like* functions:

\[
V_i(x) > 0 \quad \text{for} \quad x \in X_i
\]

\[
\frac{\partial V_i(x)}{\partial x} f_i(x) < 0 \quad \text{for} \quad x \in X_i
\]

where \( X_i \) contains all \( x \) for which submodel \( f_i \) can be activated.

- Can weaken requirement that \( V_i \) should decrease along trajectories of \( f_i \)

See the references for details and precise statements.
Summary

A whirlwind tour:
• selected results on stability and stabilization of hybrid systems

Three specific problems
• Guaranteeing stability independent of switching signal
• Design a stabilizing switching strategy (stabilizability)
• Prove stability for a given switching strategy

Focus has been on Lyapunov-function techniques
• Alternative approaches exist!

Strong theoretical results, but hard to apply in practice
• Can be overcome by developing automated numerical techniques (Lecture 2!)

References


Stability and stabilization of hybrid systems

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Part II – Computational tools

- Piecewise linear systems
- Well-posedness and solution concepts
- Linear matrix inequalities
- Piecewise quadratic stability
- Extensions
Computational stability analysis: philosophy

**Aim:** develop analysis tools that
- are computationally efficient (e.g. run in polynomial time)
- work for *most* practical problem instances
- produce guaranteed results (when they work)

![Diagram of system description and computer program leading to stable or inconclusive results.]

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**Piecewise linear systems**

**Piecewise linear system:**

1. a subdivision of $\mathbb{R}^n$ into regions $X_i$
   $$\bigcup_{i=1}^{M} X_i \subseteq \mathbb{R}^n$$
   we will assume that $X_i$ are polyhedral and disjoint (i.e. that cells only share common boundaries)

2. (possibly different) affine dynamics in each region
   $$\begin{align*}
   \dot{x}(t) &= A_i x(t) + a_i + B_i u(t) \\
   y(t) &= C_i x(t) + c_i + D_i u(t)
   \end{align*}
   \text{ for } x(t) \in X_i \quad i \in I$$
Example

Saturated linear system: \( \dot{x} = Ax + b \text{sat}(v), \quad v = k^T x \)

Three regions: negative saturation, linear operation, positive saturation

\[
\dot{x} = \begin{cases} 
    Ax - b & x \in X_1 \\
    (A - bk^T)x & x \in X_2 \\
    Ax + b & x \in X_3 
\end{cases}
\]

Cells are polyhedral (i.e., can be described by a set of linear inequalities)

Well-posedness and solutions

Definition. Let \( x(t) \subset \bigcup_{i \in I} X_i \) be an absolutely continuous function. We say that \( x(t) \) is a trajectory of the system

\[
\begin{align*}
\dot{x}(t) &= A_i x(t) + a_i + B_i u(t) \\
y(t) &= C_i x(t) + c_i + D_i u(t)
\end{align*}
\]

for \( x(t) \in X_i \) \( i \in I \)

on \([t_0, t_f]\) if, for allmost all \( t \in [t_0, t_f] \), the equation \( \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \) holds for all \( i \) with \( x(t) \subset X_i \).
Trajectories: existence and uniqueness

**Observation:** trajectories may not be unique, or may not exist.

**Example:**

\[
\begin{align*}
\dot{x}_1 &= -2x_1 - 2x_2 \text{sgn}(x_1) \\
\dot{x}_2 &= x_2 + 4x_1 \text{sgn}(x_1)
\end{align*}
\]

Initial values in \( S_1^- = \{x \mid x_1 = 0 \land x_2 \leq 0\} \) create non-unique trajectories.

Trajectories that reach \( S_1^+ = \{x \mid x_1 = 0 \land x_2 \geq 0\} \) cannot be continued.

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Attractive sliding modes

Would like to single out situations with non-existence of solutions.

**Definition.** The system

\[
\begin{align*}
\dot{x}(t) &= A_ix(t) + a_i + B_iu(t) \\
y(t) &= C_ix(t) + c_i + D_iu(t)
\end{align*}
\]

for \( x(t) \in X_i \quad i \in I \)

*is said to have an attractive sliding mode at* \( x_s \) *if there exists a trajectory with final state* \( x_s \) *but no trajectory with initial state* \( x_s \).
Generalized solutions

Solution concepts for systems with sliding modes typically averages dynamics in neighboring cells.

Note: Filippov solutions may remain on cell boundaries, and are not necessarily unique.

Equivalent dynamics on sliding modes

Example: Piecewise linear system

\[
\begin{align*}
\dot{x}_1 &= -2x_1 - 2x_2 \text{sgn}(x_1) \\
\dot{x}_2 &= x_2 + 4x_1 \text{sgn}(x_1)
\end{align*}
\]

on \( S_1^+ = \{ x \mid x_1 = 0 \land x_2 \geq 0 \} \)

Filippov solutions satisfy \( \dot{x}(t) \in \alpha A_1 x(t) + (1 - \alpha) A_2 x(t) \) for some \( \alpha \in [0, 1] \)

If \( x(t) \) should stay on \( S_1^+ \), we must have \( \dot{x}_1(t) = 0 \), i.e.,

\[
\alpha \cdot 2x_2 + (1 - \alpha) \cdot (-2x_2) = x_2(4\alpha - 2) = 0
\]

The only solution is given by \( \alpha = 1/2 \), resulting in the unique sliding dynamics

\[
\begin{align*}
\dot{x}_1 &= 0, \\
\dot{x}_2 &= x_2
\end{align*}
\]
Non-uniqueness of sliding dynamics

Observation: sliding dynamics on intersecting boundaries often non-unique

Example:

\[
\begin{align*}
\dot{x}_1 &= x_2 - \text{sgn}(x_2) \\
x_2 &= x_3 - \text{sgn}(x_2) \\
x_3 &= -2x_1 - 4x_2 - 4x_3 - x_3\text{sgn}(x_2)\text{sgn}(x_1 + 1)
\end{align*}
\]

Filippov solutions on \(S_{12} = \{x \mid x_1 = 0 \land x_2 = 0 \land |x_3| \leq 1\}\) are not unique.
(can you explain why?)

Valid Filippov solutions on \(S_{12}\) differ in time constants of a factor four or more.

Establishing attractivity of sliding modes

Note: non-trivial to detect that a pwl system has attractive sliding modes

Example: The piecewise linear system

\[
\begin{align*}
\dot{x}_1 &= -\text{sgn}(x_1) + 2\text{sgn}(x_2) \\
\dot{x}_2 &= -2\text{sgn}(x_1) - \text{sgn}(x_2)
\end{align*}
\]

has a sliding mode at the origin.

However, determining that it is attractive is not easy
- Vector field inspection or quadratic Lyapunov functions can’t be used (why?)
- Finite-time convergence to the origin can be established by noting that

\[
\frac{d}{dt}(|x_1| + |x_2|) = -2
\]
Key points

Piecewise linear systems: polyhedral partition and locally affine dynamics

\[
\begin{align*}
\dot{x}(t) &= A_i x(t) \vert a_i \vert B_i u(t) \\
y(t) &= C_i x(t) + c_i + D_i u(t)
\end{align*}
\text{for } x(t) \in X_i \quad i \in I
\]

For general piecewise linear systems, solution concepts are non-trivial
- Trajectories may not be unique, or may not exist (unless continuous)
- Meaningful solution concepts for attractive sliding modes exist (e.g. Filippov solutions)

Introducing “new modes” on cell boundaries with sliding dynamics not easy
- Sliding modes may occur on any intersection of cell boundaries
- Hard to determine if potential sliding mode is attractive
- Dynamics of sliding modes may be non-unique and non-linear

Part II – Computational tools

- Piecewise linear systems
- Well-posedness and solution concepts
- Linear matrix inequalities
- Piecewise quadratic stability
- Extensions
Linear matrix inequalities

**Linear matrix inequality (LMI):** An inequality on the form

\[ F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0 \]

where \( F_i \) are symmetric matrices, \( X > 0 \) denotes that \( X \) is positive definite.

**Example:** The condition \( P > 0 \) on standard form:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \oplus \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \oplus \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} > 0
\]

**LMI features**

- Optimization under LMI constraints is a *convex* optimization problem
  - Strong and useful theory, e.g. duality
    (we have already used it once – when?)

- Multiple LMIs is an LMI
  - Example: Lyapunov inequalities \( P > 0, A^T P + PA < 0 \) equivalent to single LMI
    \[
    \begin{bmatrix}
P & 0 \\
0 & -A^T P - PA
\end{bmatrix} > 0
    \]

- Efficient software and convenient user interfaces publicly available
  - Example: YALMIP interface by J. Löfberg at ETHZ

- S-procedure, Shur complements, ... and much more!
Example: Quadratic stabilization

Recall from Lecture 1 that $V(x) = x^T P x$ guarantees that

$$\dot{x}(t) = A_i(t)(x(t))$$

is GAS for all switching signals $i(t)$ (i.e., GUAS) if $P$ satisfies

$$P > 0$$

$$A_i^T P + PA_i < 0 \quad \forall i \in \{1, 2, \ldots, M\}$$

an LMI condition!

Consequence: quadratic Lyapunov function found efficiently (if it exists)!

Quadratic stability of PwL systems

$V(x) = x^T P x$ is a Lyapunov function for the piecewise linear system

$$\dot{x} = A_i x \quad x \in X_i$$

if we have

$$x^T P x > 0 \quad \forall x \neq 0$$

$$x^T (A_i^T P + PA_i) x < 0 \quad \forall x \in X_i \setminus 0$$

Note: not necessary to require that $A_i^T P + PA_i < 0$

How can we bring the restricted conditions into the LMI framework?
S-procedure

When does it hold that, for all $x$,

$$x^T R x \geq 0 \Rightarrow x^T P x \geq 0$$

(i.e., non-negativity of quadratic form $x^T R x$ implies non-neagivity of $x^T P x$.)

**Simple condition:** there exists $\tau \in \mathbb{R}_+$ satisfying the LMI $P \succeq \tau R$

**Extension to multiple quadratic forms:** if there exist $\tau_i \geq 0$ such that

$$P - \sum_i \tau_i R_i \succeq 0$$

then $(x^T R_1 x \geq 0) \land (x^T R_2 x \geq 0) \land \cdots \Rightarrow x^T P x \geq 0$

(non-trivial fact: simple condition is necessary if there exists $u$: $u^T R u > 0$)

Bounding polyedra by quadratic forms

**Example:** The polyhedron

$$X = \{ x \mid ||x|| \leq 1 \} = \{ x \mid (x \geq -1) \land (x \leq 1) \} = \{ x \mid (x + 1 \geq 0) \land (1 - x \geq 0) \}$$

can be described by the quadratic form

$$q(x) = \tau (x + 1)(1 - x) = \tau (1 - x^2) \geq 0$$

for $\tau \geq 0$

**In general:** for polyhedra $X_i = \{ x \mid E_i x + c_i \geq 0 \}$ the quadratic form

$$q(x) = \left( [E_i \ c_i] \begin{bmatrix} x \\ 1 \end{bmatrix} \right)^T U_i \left( [E_i \ c_i] \begin{bmatrix} x \\ 1 \end{bmatrix} \right) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T E_i^T U_i E_i \begin{bmatrix} x \\ 1 \end{bmatrix}$$

is non-negative for all $x \in X_i$ if $W_i$ has non-negative entries
Quadratic stability cont’d

Consider the piecewise linear system

\[ \dot{x} = A_i x \quad \text{for } x \in X_i = \{ x \mid E_i x \geq 0 \} \]

(no affine terms, all regions contain the origin). Then,

\[ A_i^T P + P A_i + F_i^T U_i F_i < 0 \]

**Theorem.** If there exists a positive definite matrix \( P \) and matrices \( U_i \) with non-negative entries such that

then every Filippov solution tends to zero exponentially.

---

Example

Recall the switched system

\[ \dot{x} = A_1 x \text{ for } x_1 x_2 \geq 0, \quad \dot{x} = A_2 x \text{ for } x_1 x_2 \leq 0 \]

with

\[ A_1 = \begin{pmatrix} -0.1 & 1 \\ -10 & -0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.1 & 10 \\ -1 & -0.1 \end{pmatrix} \]

from Lecture 1. Applying the above procedure, we find

\[ P = I, \text{ e.g., } V(x) = x^T x. \]

(stability cannot be verified without S-procedure – can you explain why?)
Piecewise quadratic Lyapunov functions

Natural to consider continuous, *piecewise quadratic*, Lyapunov functions

\[ V(x) = x^T P_i x + 2q_i^T x + r_i = \begin{bmatrix} x^T \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad \text{for } x \in X_i \]

Surprisingly, such functions can also be computed via optimization over LMIs.

Relation to multiple Lyapunov functions:

- Local expressions for \( V(x) \) are Lyapunov-like functions for associated dynamics (stronger relationship will emerge in the extensions)

Convenient notation

Use the augmented state vector

\[ \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \]

and re-write system dynamics as

\[ \begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} a_i & a_i & B_i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ u \end{bmatrix} + \begin{bmatrix} A_i \\ C_i \\ D_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \]

When analyzing properties of the equilibrium \( x = 0 \) we let

- \( I_0 \subseteq I \) be the set of indices for regions containing origin
- \( I_1 \subseteq I \) be the set of indices for regions that do not contain origin

and assume that \( a_i = c_i = 0 \) for \( i \in I_0 \)
Enforcing continuity

How to ensure that the Lyapunov function candidate

\[ V(x) = \begin{bmatrix} x^T & q_i \end{bmatrix} \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{x}^T P_i \bar{x} \]

for \( x \in X_i \)

is continuous across cell boundaries?

Proposition. \( \bar{x}^T P_i \bar{x} = \bar{x}^T P_j \bar{x} \) for all \( x \in X_i \cap X_j = \{ x \mid \bar{h}_{ij}^T \bar{x} = 0 \} \)
if and only if there exists \( t_{ij} \in \mathbb{R}^{n+1} \) such that

\[ P_i = P_j + \bar{h}_{ij}^T t_{ij} + t_{ij}^T \bar{h}_{ij} \]

Enforce one linear equality for each cell boundary.

Enforcing continuity (II)

Alternative: direct parameterization

For each region, construct continuity matrices \( \bar{F}_i = [F_i \quad f_i] \) such that

\[ \bar{F}_i \bar{x} = \bar{F}_j \bar{x} \] for all \( x \in X_i \cap X_j \)

and consider Lyapunov functions on the form

\[ V(x) = \bar{x}^T \bar{F}_i^T T \bar{F}_i \bar{x} \] for \( x \in X_i \)

(the free variables are now collected in the symmetric matrix \( T \))

To make Lyapunov function quadratic in regions that contain origin, we also require

\[ f_i = 0 \] for \( i \in I_0 \)

(construction automated in, for example, Pwltools)
Piecewise quadratic stability

Theorem (Piecewise Quadratic Stability). Consider symmetric matrices $T$, $U_i$ and $W_i$ such that $U_i$ and $W_i$ have nonnegative entries, while $P_i = F_i^T T F_i$ and $P_i = F_i^T T F_i$ satisfy

$$
\begin{align*}
0 &> A_i^T P_i + P_i A_i + E_i^T U_i E_i \\
0 &< P_i - E_i^T W_i E_i
\end{align*} \quad i \in I_0
$$

and

$$
\begin{align*}
0 &> \overline{A_i}^T P_i + P_i \overline{A_i} + \overline{E_i}^T U_i E_i \\
0 &< P_i - \overline{E_i}^T W_i \overline{E_i}
\end{align*} \quad i \in I_1
$$

Then every trajectory $x(t) \in \bigcup_{i \in I} X_i$ satisfying

$$
\dot{x} = A_i x + a_i \quad \text{for } x \in X_i
$$
tends to zero exponentially.

Example

Piecewise linear system with partition shown below,

$$
A_1 = A_3 = \begin{bmatrix} c & \omega \\ -\alpha \omega & -\epsilon \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} c & \alpha \omega \\ -\omega & -\epsilon \end{bmatrix},
$$

and $\alpha = 5$, $\omega = 1$, $\epsilon = 0.1$

(Clearly) not quadratically stable, but pwQ Lyapunov function readily found.
Potential sources of conservatism

1. Quadratic Lyapunov functions necessary and sufficient for linear systems, but piecewise quadratic Lyapunov functions not necessary for stability of PWL systems.

2. S-procedure terms $\bar{E}_i^T W_i \bar{E}_i$ effectively the sum of several quadratic forms

$$\bar{x}^T \bar{E}_i^T W_i \bar{E}_i \bar{x} = \sum_i \sum_j w_{ij} (\bar{c}_i^T \bar{x})^T (\bar{c}_j^T \bar{x})$$

hence, S-procedure is not guaranteed to be loss-less (better tools exist)

3. Use of affine terms and strict inequalities can also be conservative.

Extensions

Many extensions possible:
• determining regions of attraction (i.e. non-global stability properties)
• Lyapunov functions that guarantee stability of potential sliding modes
• nonlinear and uncertain dynamics in each region
• performance analysis (e.g. $L_2$-gains)
• (some) control synthesis
• hybrid systems (overlapping regions) and discontinuous Lyapunov fnics.
• Lyapunov functionals and Lagrange stability
• stability of limit cycles
• similar tools for discrete-time hybrid systems

(too much to be covered in this lecture!)

We will sketch a couple of extensions
Performance analysis

Theorem (Upper Bound on $L_2$ Gain). Suppose there exist symmetric matrices $T$, $U_i$ and $W_i$ such that $U_i$ and $W_i$ have non-negative entries, while $P_i - F_i^TTF_i$ and $P_i - F_i^TTF_i$ satisfy

$$0 > \begin{bmatrix} P_iA_i + A_i^TP_i + C_i^TC_i + F_i^TU_iF_i & P_iB_i \\ R_i^TP_i & -\gamma^2 I \end{bmatrix} \quad \text{for } i \in I_0$$

$$0 > \begin{bmatrix} P_i\overline{A}_i + \overline{A}_i^TP_i + \overline{C}_i^T\overline{C}_i + \overline{F}_i^TU_i\overline{B}_i & \overline{P}_i\overline{B}_i \\ \overline{R}_i^T\overline{P}_i & -\gamma^2 I \end{bmatrix} \quad \text{for } i \in I_1$$

Then for every trajectory with $x(0) = 0$, $\int_0^\infty (\|x\|_2^2 + \|u\|_2^2) \, dt < \infty$

$$\int_0^\infty \|u\|_2^2 \, dt < \gamma^2 \int_0^\infty \|u\|_2^2 \, dt.$$ 

The best upper bound on the $L_2$ induced gain is achieved by minimizing $\gamma^2$ subject to the constraints defined by the inequalities.

Proof. Pre/postmultiply with $(x, u)$, note that LMIs imply dissipation inequality

Example

Saturated linear system (unit saturation)

$$G_1(s) = \frac{s - 3}{16s^2 + s + 2}$$

$$G_2(s) = \frac{s + 7}{4s^2 + 3s + 12}$$

Quadratic storage functions fail to bound $L_2$-gain.

Piecewise quadratic storage function yields bounds

$$5.52 \leq \gamma \leq 5.54$$
Linear hybrid dynamical systems

Linear hybrid dynamical system (LHDS)
\[
\begin{align*}
\dot{x}(t) &= A_i(t)x(t) + a_i(t) \\
\nu(t^+) &= \nu(x(t), i(t))
\end{align*}
\]
\(\nu\) described by finite automaton whose state changes when \(x\) hits transition surfaces

\[
S_{ij} = \{x \mid \bar{f}_{ij} \bar{x} = 0\}
\]
and for each \(i\), the feasible \(x\) bounded by a polyhedron \(X_i = \{x \mid E_i \bar{x} \succeq 0\}\)

Discontinuous Lyapunov functions

Multiple quadratic (discontinuous, pwq) Lyapunov function via LMIs

Theorem. Consider symmetric matrices \(U_i, W_i\) with non-negative entries, symmetric matrices \(P_i, \bar{P}_i\), and vectors \(t_{jk}, \bar{t}_{jk}\) such that

\[
\begin{align*}
0 > A_i^T P_i + P_i A_i + F_i^T U_i F_i & \quad i \in I_0 \\
0 < P_i - E_i^T W_i E_i & \quad i \in I_0
\end{align*}
\]

\[
\begin{align*}
0 > A_i^T \bar{P}_i + \bar{P}_i A_i + \bar{F}_i^T U_i \bar{F}_i & \quad i \in I_1 \\
0 < \bar{P}_i - E_i^T \bar{W}_i E_i & \quad i \in I_1
\end{align*}
\]

\[
0 < P_j - \bar{P}_k + \bar{f}_{jk} \bar{t}_{jk}^T + t_{jk} f_{jk}^T & \quad (j, k) \in T, \quad j \in I_1 \text{ or } k \in I_1
\]

\[
0 < P_j - P_k + f_{jk} t_{jk}^T + t_{jk} f_{jk}^T & \quad (j, k) \in T, \quad j, k \in I_0
\]

Then every trajectory of the LHDS tends to zero exponentially.

Note: conditions \((3,4)\) imply that \(V(t)\) decreases at points of discontinuity.
Example

Linear hybrid system
\[ \dot{x}(t) = A_i(t)x(t) \]
\[ i(t^+) = \begin{cases} 2 & \text{if } i(t) = 1 \text{ and } x_2 = -10x_1 \\ 1 & \text{if } i(t) = 2 \text{ and } x_2 = 2x_1 \end{cases} \]
with
\[ A_1 = \begin{pmatrix} -1 & -100 \\ 10 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 10 \\ -100 & 1 \end{pmatrix} \]

Trajectories and multiple Lyapunov function found by LMI formulation

Discrete-time versions

Discrete-time piecewise linear systems
\[ x[k + 1] = A_i x[k] + a_i + B_i u[k] \]
x[k] ∈ X_i
and piecewise quadratic Lyapunov (not necessarily continuous) functions
\[ V(x[k]) = x[k]^T P_i x[k] + 2q_i x[k] + r_i \]
x[k] ∈ X_i

We have
\[ \Delta V(x[k]) = V(x[k] | 1]) - V(x[k]) = x[k]^T A_i^T P_j A_i x[k] + 2q_i^T P_i a_i + 2q_j^T a_j + r_j - x[k]^T P_i x[k] + 2q_i^T x[k] + r_i \]
\[ = \begin{pmatrix} x[k] \\ 1 \end{pmatrix}^T \begin{pmatrix} A_i^T P_j A_i - P_i \\ a_i^T P_i a_i + 2q_j a_j + r_j - r_i \end{pmatrix} \begin{pmatrix} x[k] \\ 1 \end{pmatrix} \]

for \( x[k] ∈ X_{ij} = \{ x \mid x \in X_i ∧ A_i x + a_i ∈ X_j \} = \{ x \mid \bar{E}_{i} x ≥ 0 ∧ \bar{E}_{i} A_i x ≥ 0 \} \)
Discrete-time versions

Discrete-time globally asymptotically stable if there exist matrices $P_i, q_i, r_i, U_{ij}$ where $W_{ij}$ has non-negative entries, and a non-negative scalar $\varepsilon > 0$, such that

$$
\begin{bmatrix}
\Lambda_i^T P_j A_i - P_i \\
\lambda_i^T P_j a_i + q_j \\
\end{bmatrix}
\begin{bmatrix}
\Lambda_i^T P_j a_i + q_i \\
\end{bmatrix}
\begin{bmatrix}
\lambda_j^T P_i a_j + 2q_j a_i + r_j - r_i \\
\end{bmatrix}
+ \bar{E}_{ij} U_{ij} \bar{E}_{ij} \leq
\begin{bmatrix}
-\varepsilon I & 0 \\
0 & 0 \\
\end{bmatrix}
$$

(note: in most solvers, you will need to treat $X_i, i \in I_0$ separately)

Observations:
- Again, LMI conditions, hence efficiently verified!
- Potentially one LMI for every pair (i,j) of modes.

Comparison with alternatives

Biswas et al. generated optimal hybrid controllers for randomly generated linear systems, and compared performance of several computational methods

Typical results:

<table>
<thead>
<tr>
<th>Method</th>
<th>Success</th>
<th>Solution Time</th>
<th>Setup Time</th>
<th>Success</th>
<th>Solution Time</th>
<th>Setup Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadratic</td>
<td>45/50</td>
<td>0.7 sec.</td>
<td>0.4 sec.</td>
<td>43/50</td>
<td>1.1 sec.</td>
<td>0.5 sec.</td>
</tr>
<tr>
<td>Piecewise Quadratic</td>
<td>50/50</td>
<td>0.7 sec.</td>
<td>1.3 sec.</td>
<td>50/50</td>
<td>1.9 sec.</td>
<td>2.5 sec.</td>
</tr>
<tr>
<td>Common SOS order 4</td>
<td>42/50</td>
<td>7.6 sec.</td>
<td>82.2 sec.</td>
<td>32/50</td>
<td>11.4 sec.</td>
<td>141.9 sec.</td>
</tr>
<tr>
<td>Piecewise SOS order 4</td>
<td>35/50</td>
<td>12.1 sec.</td>
<td>100.0 sec.</td>
<td>31/50</td>
<td>80.6 sec.</td>
<td>263.8 sec.</td>
</tr>
</tbody>
</table>

Table 2. The number of regions were between 9 and 15 with 9-47 transitions.

Very strong performance, but computational effort increases rapidly
Summary

Computational tools for stability analysis of one class of hybrid systems

Piecewise linear systems
- Partition of state space into polyhedra with locally affine dynamics
- Solution concepts: trajectories and Flippov solutions
- Given a pwli model, it is non-trivial to detect attractive sliding modes

Piecewise quadratic Lyapunov functions
- Efficiently computed via optimization over linear matrix inequalities
- Potentially conservative, but strong practical performance

Many extensions, but much work remains!

References


Stability and stabilization of hybrid systems

Mikael Johansson
School of Electrical Engineering
KTH

Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system
Constrained control via min-max selectors

Common “pre-HYCON” approach for constrained control

**Aim:** tracking primary variable (y), while keeping secondary variable (z) within limits

![Diagram of constrained control system](image)

**Numerical example**

Specific example with

\[ G_1(s) = \frac{40}{0.5s^7 + 2s^5 + 22s + 40} \quad G_2(s) = \frac{5}{s^5 + 7s + 5} \quad C(s) = \frac{s^2 + 3s + 3}{0.02s^2 + s + 0.01} \]

and proportional constraint controllers.

Control without constraint handling  Control with constraint handling
A loop transformation

Linear system interconnected with 3-input/1-output nonlinearity

Loop transformation reduces dimension of nonlinearity by one:

still, few techniques apply to such systems
(e.g. small gain and LDI do not work)

Stability analysis

However, nonlinearity (and hence system) is piecewise linear:

LMI computations return quadratic Lyapunov function
(but S-procedure needed)
Part III – Examples

- Constrained control via min-max selectors
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Fed-batch cultivation of \textit{E. coli}

Recombinant (genetically modified) \textit{E. coli} bacteria used to produce proteins.

Bioreactor control: Add feed (nutrition) and oxygen to maximize cell growth.

Fed-batch: feed added continuously, at limiting rate

[Velut, 2005]
Control objective

Objective: maximize feed rate while ensuring that
- oxygen level does not drop too low (acetate production, inhibited growth)
- glucose is not in excess (“overflow metabolism”)

Probing control

Control strategy: increase feed while no acetate formed, decrease otherwise

Acetate formation detected by probing:
- add pulse in feed, observe if oxygen consumed
A piecewise linear abstraction

Simplified model of reactor dynamics
\[ \dot{x} = ax + bf(v) \]
\[ y = cx \]
where \( f(v) \) is a piecewise linear function
\[ v(t) = u_k + u_p(t) \quad t \in [kT, (k + 1)T] \]

Integrating the response over a pulse period, we find the discrete-time model
\[ x[k + 1] = Ax[k] + B \left[ \frac{f(u_k)}{f(u_k + u_p^0)} \right] \]
\[ y[k] = C'r[k] + D \left[ \frac{f(u_k)}{f(u_k + u_p^0)} \right] \]
Piecewise linear if \( u_k \) is a linear in \( x \).

Control strategy

Assume a linear integral control
\[ u[k + 1] = u[k] + K(y_{ref}[k] - y[k]) \]
fixed length of probing cycle \( T \) and probing pulse \( T - T_c \)

To model saturation in glucose uptake, consider
\[ f(v) = \min(v, r^*) \]
This results in a piecewise linear systems with three regions (why not two?)

Control objective is now to drive system towards saturation.
Control to saturation

The formulation in Lecture 2 does not return any feasible solution
• integrator dynamics in unbounded regions $\Rightarrow$ not exponentially stable

Two potential approaches:
• Prove convergence for initial values within (large but bounded) region
  (can be done by adding S-procedure terms)
• Remove implicit equality constraints by state-transformation
  (more satisfying, but more complex; see Velut)

With modifications, stability can (often) be proven VIA pwq Lyapunov fncs.

Numerical results

Stability regions for one specific problem instance (reactor parameters)
• red dots bound region where stability can be established numerically
• shaded regions are shown to be unstable (via local analysis)
Performance analysis

Stability often not enough with stability – would like to optimize performance
• for example, the ability to track time-varying saturation level

Can compute bound $\gamma$ on performance

$$\sum_k (x[k] - x_{ref}[k])^T \tilde{Q}(x[k] - x_{ref}[k]) \leq \gamma^2 \sum_k (r[k + 1] - r[k])^2$$

for all reference trajectories $r[k]$ via LMI computations.

Note: typically large system descriptions...

Numerical example

Simulations for specific $r[k]$  \hspace{2cm} $\gamma$ for all rate-limited references

Parameter contours suggest optimal parameters

$K \approx 1.4, \ T_c \approx 1.3$
Tuning rules

Similar behavior observed for various parameter values of the process.

Based on this observation, Velut suggests the following tuning rules

\[ K = \frac{1}{\sigma(T - T_c) - \sigma(T - T_c)} \]
\[ y_{\text{ref}} = \frac{\sigma(T - T_c) - \sigma(T - T_c)}{2} n_p^c \]
\[ 1 < aT_c < 2 \]

where \( \sigma(t) \) is the unit step response of the linear dynamics.

Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system
A simple model for car dynamics

Simple model:

\[ M \ddot{v} = F - F_i \quad \text{car dynamics} \]
\[ F_i = kn^2 \text{sign}(v) - Mg \text{sin} \alpha \quad \text{load force} \]
\[ F = \frac{p}{rT} \quad \text{gear box relations} \]
\[ v = \frac{r}{pr} \quad \text{gear box relations} \]

Inputs: motor torque \( T \) and road incline \( \alpha \); output \( \omega \)

\[ \dot{v} = \frac{1}{M} T u - \frac{k}{M} v^2 \text{sign} v - g \text{sin} \alpha \]
\[ \omega = vu \]

where \( u = \frac{p}{r} \) is the discrete input, determined by the current gear

To emphasize this dependence, we write

\[ u = u_i := \frac{p_i}{r_i} \quad \text{when using gear } i \quad [\text{Pettersson, 1999}] \]

Gear-switching

Gear-switching strategy:

\[ i(t^+) = i(t) + 1 \quad \text{if } \omega > \omega_i^{hi}(t) \]
\[ i(t^+) = i(t) - 1 \quad \text{if } \omega < \omega_i^{lo}(t) \]

Can be represented by hybrid automaton with four discrete states
Torque control and bumpless transfer

Base controller: non-linear PI

\[ T = P + I + \frac{k}{u_i} v^2 \text{sign} v \]
\[ P = K_i(t)(v_{\text{ref}} - v) \]
\[ \frac{d}{dt} I = \frac{K_i(t)}{T_i(t)} (v_{\text{ref}} - v) \]

Changes in acceleration when shifting gears avoided via bumpless transfer:

\[ u_i K_i = u_j K_j \]
\[ I(t^+) = \frac{u_i(t)}{u_i(t^+)} I(t) \]

for all feasible gear changes \( i \rightarrow j \).
( compatible values of \( K_i \), changes in integral state)

Hybrid system model

Need extended hybrid model that allows for jumps in the continuous state

\[ \dot{x}(t) = f(x(t), i(t)) \]
\[ x(t^+) = \rho(x(t), i(t)) \]
\[ i(t^+) = \nu(x(t), i(t)) \]

LMI formulation possible if jump map is affine in \( x \).
Numerical example

Closed loop system is switched linear system

\[
\begin{bmatrix}
\frac{d}{dt} [e] \\
\end{bmatrix} =
\begin{bmatrix}
-u_i K_i/M & -p_i/M \\
K_i/T_i & 0 \\
\end{bmatrix}
\begin{bmatrix}
e \\
I \\
\end{bmatrix} \quad \text{for } c \in X_i
\]

where \( e = v_{\text{ref}} - v \) and

\[
u_i = \{50, 32, 20, 14\}
\]
\[
K_i = \{3.75, 5.86, 9.37, 13.39\}
\]
\[
M = 1500, \ T_i = 40, \ T_i K_i = 187.5
\]

Simulation for \( v_{\text{ref}} = 30 \)

Stability

If affine reset maps

\[
x(t^+) = H_i(t) x(t^+) \bar{x}(t)
\]

then, \( \bar{x}(t^+)^T P_j \bar{x}(t^+) < \bar{x}(t)^T P_i \bar{x}(t) \) is guaranteed by solution to LMI

\[
0 < P_i - H_{jk}^T P_k H_{jk} + \bar{f}_{jk}^T \bar{f}_{jk} + f_{jk}^T \bar{f}_{jk}
\]

Can extend discontinuous Lyapunov function computations from Lecture 2

**Gear-box example:** solution found \( \rightarrow \) exponential convergence to \( v_{\text{ref}} \)

**Remark:** analysis needs to be repeated for each value of \( v_{\text{ref}} \) (as in bioreactor example)
Part III – Examples

- Constrained control via min-max selectors
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- A simple relay system

More of a theoretical challenge…

Consider a linear control system under hysteresis relay feedback...

\[ A = \begin{pmatrix} 0.1 & 1 \\ 0 & -0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

Simulations suggest system is stable, yet no pwq Lyapunov function found.

[Hassibi, 2000]
The challenge

Q: why do piecewise quadratic methods fail, how can they be improved?

The more general challenge:

Put the methods to the test of challenging engineering problems, and help to contribute to the development of improved analysis tools!

References


