Solution Concepts and Well-posedness of Hybrid Systems

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HYCON Summer School on Hybrid Systems

Key issues:

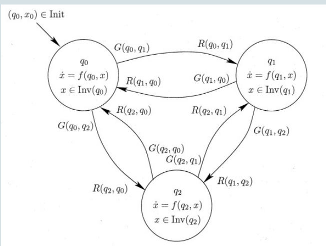
- Solution concepts
- Well-posedness: existence & uniqueness of solutions given an initial condition

Outline lecture

- Smooth systems: differential equations
- Switched systems: Discontinuous differential equations: "classics"
- Hybrid automata
- Zenoness: importance of choice of solution concept
- Some piecewise linear, linear relay and complementarity systems
- Summary

Solution concept

Description format / syntax / model
$\downarrow$
solutions / trajectories / executions/ semantics/ behavior



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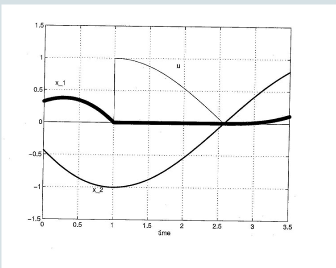


Well-posedness: given initial condition does there **exists** a solution and is it **unique**?

Let's start simple ...

Smooth differential equations

Example $\dot{x} = f(t,x) \qquad x(t_0) = x_0.$

A solution trajectory is a function $x : [t_0, t_1] \mapsto \mathbb{R}^n$ that is continuous, differentiable and satisfies $x(t_0) = x_0$ and

$$\dot{x}(t) = f(t, x(t)) \text{ for all } t \in (t_0, t_1)$$

Well-posedness: given initial condition does there **exists** a solution and is it **unique**?

Well-posedness

Example $\dot{x} = 2\sqrt{x}$, $x(0) = 0$. Solutions: $x(t) = 0$ and $x(t) = t^2$.

Local existence and uniqueness of solutions given an initial condition:

Theorem 1 Let $f(t,x)$ be piecewise continuous in t and satisfy the following Lipschitz condition: there exist an $L > 0$ and $r > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

and all x and y in a neighborhood $B := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ of x_0 and for all $t \in [t_0, t_1]$.

$$\Downarrow$$

There is a $\delta > 0$ s.t. a unique solution exists on $[t_0, t_0 + \delta]$ starting in x_0 at t_0.

Global well-posedness

Example $\dot{x} = x^2 + 1$, $x(0) = 0$. Solution: $x(t) = \tan t$. **Local** on $[0, \pi/2)$.

- Note that we have $\lim_{t \uparrow \pi/2} x(t) = \infty$. Finite escape time!

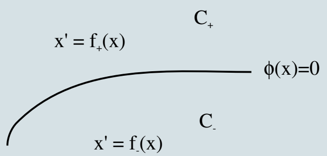
Theorem 2 (Global Lipschitz condition) Suppose $f(t,x)$ is piecewise continuous in t and satisfies

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|$$

for all x, y in $\mathbb{R}^n$ and for all $t \in [t_0, t_1]$. Then, a unique solution exists on $[t_0, t_1]$ for any initial state x_0 at t_0.

- Not necessary: $\dot{x} = -x^3$ not glob. Lipsch., but unique global solutions.
- Also in hybrid systems, but even more awkward stuff (Zeno)

Discontinuous differential equations: a class of switched systems



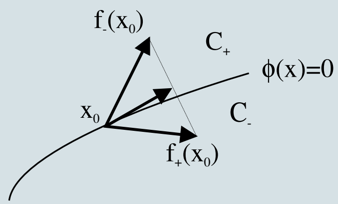
$$\dot{x} = \begin{cases} f_+(x) & \text{, if } x \in C_+ := \{x \in \mathbb{R}^n \mid \phi(x) > 0\} \\ f_-(x) & \text{, if } x \in C_- := \{x \in \mathbb{R}^n \mid \phi(x) < 0\} \end{cases}$$

- x in interior of C_- or C_+: just follow!
- $f_-(x)$ and $f_+(x)$ point in same direction: just follow!

$$n(x) = \frac{\nabla\phi(x)}{\|\nabla\phi(x)\|} \text{ then } (n(x)^T f_-(x)) \cdot (n(x)^T f_+(x)) > 0$$

- $n(x)^T f_+(x) > 0$ ($f_+(x)$ points towards C_+) and $n(x)^T f_-(x) < 0$ ($f_-(x)$ points towards C_-): At least two trajectories

Sliding modes



$n(x)^T f_+(x) < 0$ ($f_+(x)$ points towards C_-) and $n(x)^T f_-(x) > 0$ ($f_-(x)$ points towards C_+).

No classical solution

- Relaxation: spatial (hysteresis) Δ, time delay τ, smoothing ε
- Chattering / infinitely fast switching (limit case $\Delta \downarrow 0$, $\varepsilon \downarrow 0$, and $\tau \downarrow 0$)

Filippov's convex definition: convex combination of both dynamics

$$\dot{x} = \lambda f_+(x) + (1 - \lambda) f_-(x) \text{ with } 0 \leq \lambda \leq 1$$

such that x moves ("slides") along $\phi(x) = 0$. **"Third mode ... "**

Differential inclusions

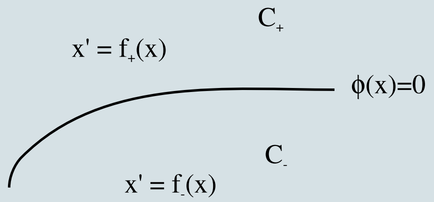
$$\dot{x} = \begin{cases} f_+(x), & \text{if } \phi(x) > 0 \\ \lambda f_+(x) + (1-\lambda) f_-(x), & \text{if } \phi(x) = 0,\ 0 \le \lambda \le 1 \\ f_-(x), & \text{if } \phi(x) < 0, \end{cases}$$

Differential inclusion $\dot{x} \in F(x)$ with set-valued

$$F(x) = \begin{cases} \{f_+(x)\}, & \phi(x) > 0 \\ \{\lambda f_+(x) + (1-\lambda) f_-(x) \mid \lambda \in [0,1]\}, & \phi(x) = 0 \\ \{f_-(x)\}, & \phi(x) < 0 \end{cases}$$

Definition 3 A function $x : [a,b] \mapsto \mathbb{R}^n$ is a *solution* of $\dot{x} \in F(x)$, if x is absolutely continuous and satisfies $\dot{x}(t) \in F(x(t))$ for almost all $t \in [a,b]$.

A well-posedness result



- f_- and f_+ are continuously differentiable (C^1)
- ϕ is C^2
- the discontinuity vector $h(x) := f_+(x) - f_-(x)$ is C^1

If for each point x with $\phi(x) = 0$ at least one of the two inequalities $n(x)^T f_+(x) < 0$ or $n(x)^T f_-(x) > 0$ (for different points a different inequality may hold), then the Filippov solutions exist and are unique.

Alternative: Utkin's equivalent control definition

$$\dot{x} = f(x,u) \text{ with } u = \begin{cases} g_+(x), & \xi(x) > 0 \\ g_-(x), & \xi(x) < 0 \end{cases}$$

- Sliding mode: $f_+(x) := f(x, g_+(x))$ and $f_-(x) := f(x, g_-(x))$ point outside C_+ and C_-, resp.

$$u_{\text{equiv}} \in U(x) := \begin{cases} \{g_+(x)\}, & \text{if } \xi(x) > 0 \\ \{\lambda g_+(x) + (1-\lambda) g_-(x) \mid \lambda \in [0,1]\}, & \text{if } \xi(x) = 0 \\ \{g_-(x)\}, & \text{if } \xi(x) < 0 \end{cases}$$

Differential inclusion

$$\dot{x} \in F(x) := f(x, U(x)) = \{f(x,u) \mid u \in U(x)\}$$

"Idealization" determines Filippov/ Utkin / different solution concept!

Example

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - u \\ \dot{x}_2 &= 2x_2(u^2 - u - 1) \\ u &= \begin{cases} 1, & \text{if } x_1 > 0 \\ -1, & \text{if } x_1 < 0. \end{cases} \end{aligned}$$

Two "original" dynamics:

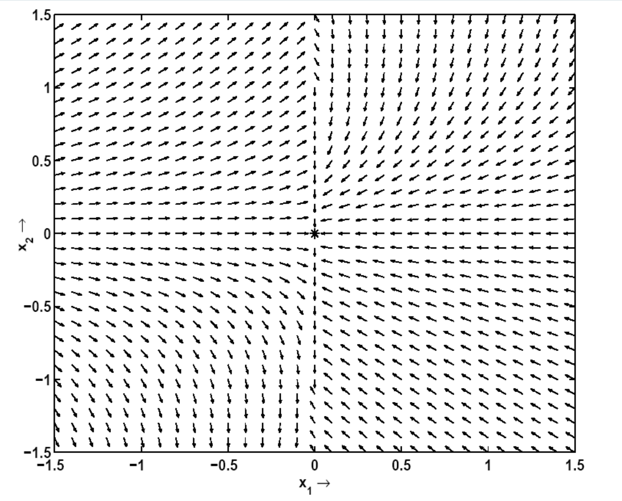
- C_+ the region $x_1 > 0$: $\dot{x} = f_+(x)$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - 1 \\ \dot{x}_2 &= -2x_2 \end{aligned}$$

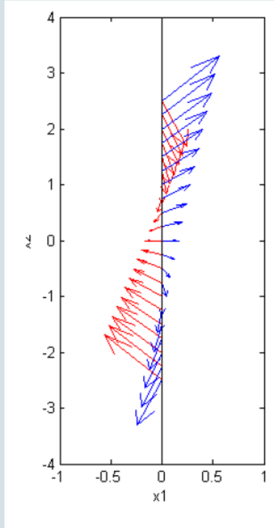
- C_- the region $x_1 < 0$ $\dot{x} = f_-(x)$

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + 1 \\ \dot{x}_2 &= 2x_2 \end{aligned}$$

Vector fields



Vector fields: zoom



Sliding modes?

Two “original” dynamics:

- C_+ the region $x_1 > 0$: $\dot{x} = f_+(x)$

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - 1\\ \dot{x}_2 &= -2x_2\end{aligned}$$

- C_- the region $x_1 < 0$: $\dot{x} = f_-(x)$

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + 1\\ \dot{x}_2 &= 2x_2\end{aligned}$$

- $n(x)^T f_+(x) = x_2 - 1 < 0 \quad \longrightarrow \quad x_2 < 1$
- $n(x)^T f_-(x) = x_2 + 1 > 0 \quad \longrightarrow \quad x_2 > -1$
- Sliding possible in $x_1 = 0$ and $x_2 \in [-1, 1]$.

Filippov’s solution concept

Two “original” dynamics:

- C_+ the region $x_1 > 0$: $\dot{x} = f_+(x)$

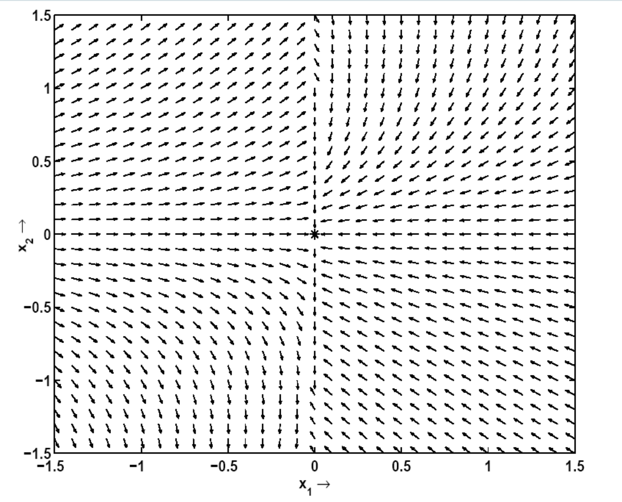
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - 1\\ \dot{x}_2 &= -2x_2\end{aligned}$$

- C_- the region $x_1 < 0$: $\dot{x} = f_-(x)$

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 + 1\\ \dot{x}_2 &= 2x_2\end{aligned}$$

- Filippov: Take convex combination of dynamics such that state slides on $x_1 = 0$: Hence, $x_1 = \dot{x}_1 = 0$.
- $\lambda(x_2 - 1) + (1 - \lambda)(x_2 + 1) = 0$ implies $\lambda = \frac{1}{2}(x_2 + 1)$
- Hence, $\dot{x}_2 = \lambda(-2x_2) + (1 - \lambda)(2x_2) = -2x_2^2$
- 0 is unstable equilibrium.

Vector fields: Filippov's case



Utkin's solution concept

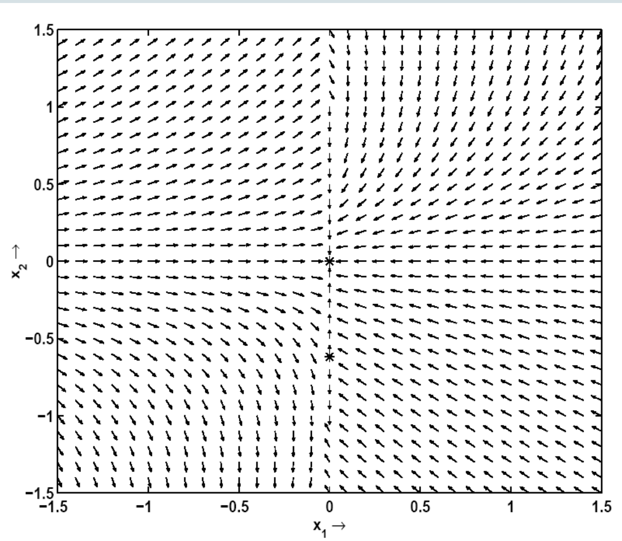
$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - u\\ \dot{x}_2 &= 2x_2(u^2 - u - 1)\\ u &= \begin{cases} 1, & \text{if } x_1 > 0\\ -1, & \text{if } x_1 < 0.\end{cases}\end{aligned}$$

- The equivalent control u_{equiv} is such that state slides along $x_1 = 0$. Hence, $x_1 = \dot{x}_1 = 0$ and thus $u_{\text{equiv}} = x_2$ and

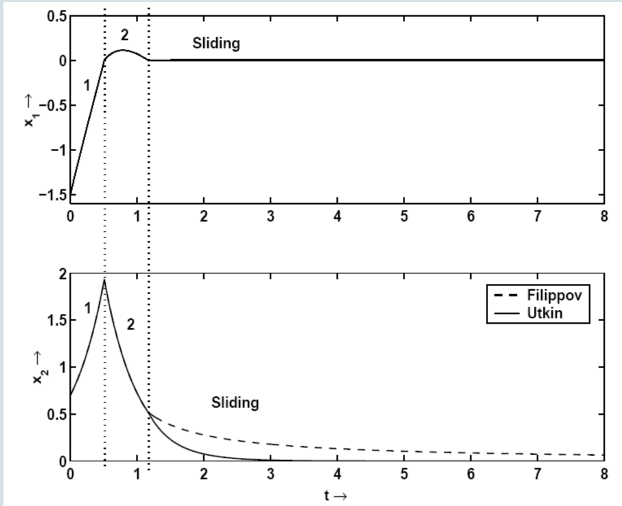
$$\dot{x}_2 = 2x_2(x_2^2 - x_2 - 1)$$

- Equilibria: -0.618 (unstable) and 0 (stable)

Vector fields: Utkin's case

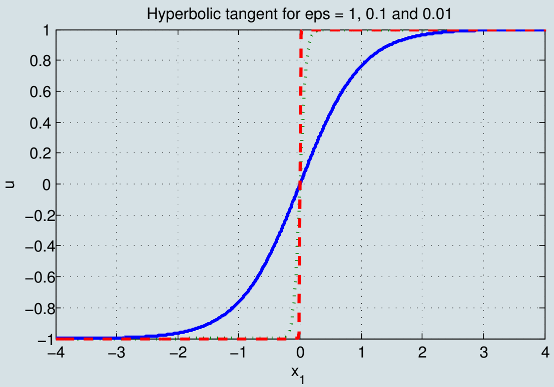


Solution trajectories



Two relaxations

- **Smoothing** $u = \tanh(x_1/\varepsilon)$
- **hysteresis type of switching** parameter Δ



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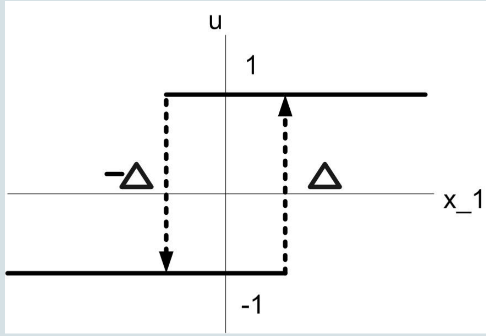
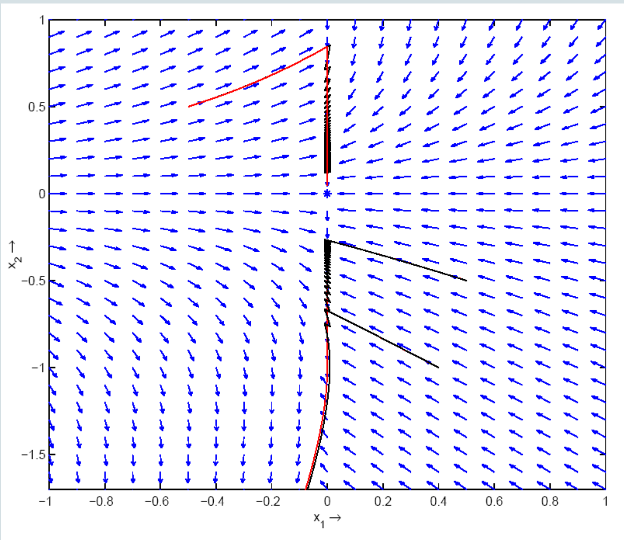
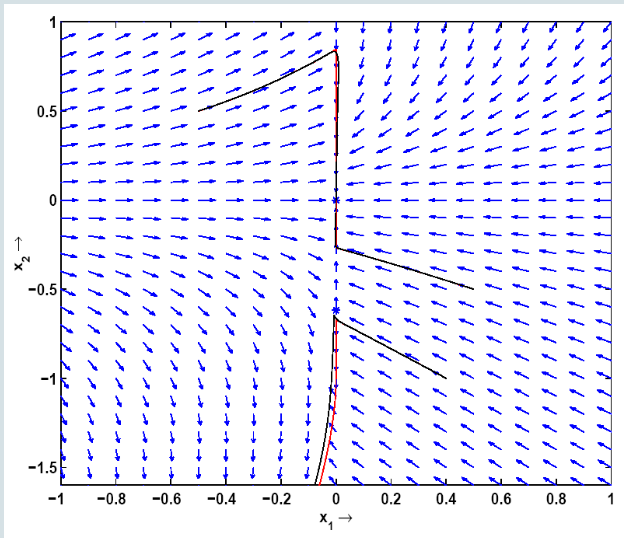


Solution trajectories: Filippov's case + hysteresis



Solution trajectories: Utkin's case + smoothing



Conclusions on discontinuous dynamical systems

- Two mathematical solutions concepts: Filippov + Utkin
- Both limit cases ("idealizations") of very fast switching
- Which one you use depends on non-ideal cases (regularizations)
- Sliding mode might be seen as third mode in hybrid automaton. Some subtleties in HA solution concept!

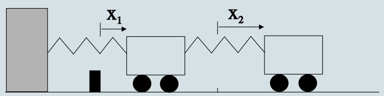
From classical to modern solution concepts

Hybrid Systems

- Smooth phases (governed by differential equations)
- Discrete events and actions

Smooth phases separated by event times ...

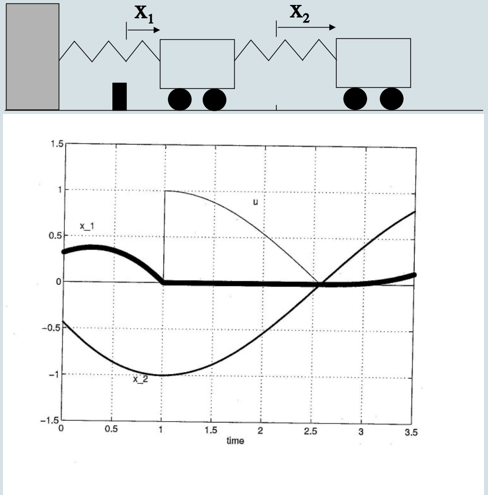
Event times



$$
\begin{aligned}
\dot{x}_1(t) &= x_3(t)\\
\dot{x}_2(t) &= x_4(t)\\
\dot{x}_3(t) &= -2x_1(t) + x_2(t) + z(t)\\
\dot{x}_4(t) &= x_1(t) - x_2(t)\\
w(t) &= x_1(t)\\
w(t) &\geq 0,\ z(t) \geq 0,\ \{w(t) = 0 \text{ or } z(t) = 0\}
\end{aligned}
$$

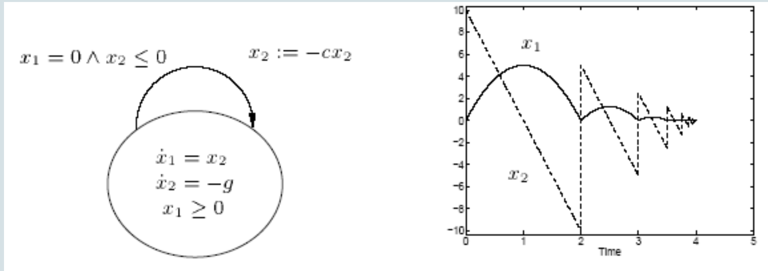
unconstrained	constrained
$\dot{x}_1(t) = x_3(t)$	$\dot{x}_1(t) = x_3(t)$
$\dot{x}_2(t) = x_4(t)$	$\dot{x}_2(t) = x_4(t)$
$\dot{x}_3(t) = -2x_1(t) + x_2(t)$	$\dot{x}_3(t) = -2x_1(t) + x_2(t) + z(t)$
$\dot{x}_4(t) = x_1(t) + x_2(t)$	$\dot{x}_4(t) = x_1(t) + x_2(t)$
$z(t) = 0$	$w(t) = x_1(t) = 0.$

unconstrained	constrained
$w(t) \geq 0$	$z(t) \geq 0$



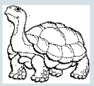
- *Event times set* $\mathcal{E}$ is $\{0, 1, 1 + \frac{\pi}{2}\}$

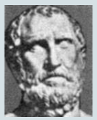
Example: Bouncing ball



- Reset $x_2(\tau+) := -cx_2(\tau-)$ when $x_1(\tau-) = 0$ and $x_2(\tau-) \le 0$
- The event times: $\tau_{i+1} = \tau_i + \frac{2c^i x_2(0)}{g}$ when $x_1(0) = 0$ and $x_2(0) > 0$.
- $\lim_{i \to \infty} \tau_i = \tau^* = \frac{2x_2(0)}{g - gc} < \infty$

Zeno of Elea and one of his paradoxes





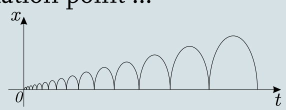
Distance Travelled (m) by Achilles	Event times of A reaching previous T position
1	1
0.5	1.5
0.25	1.75
0.125	1.875
0.0625	1.9375
0.03125	1.96875
0.015625	1.984375
0.0078125	1.9921875
0.00390625	1.99609375
0.001953125	1.998046875

Definition 4 A set $\mathcal{E} \subset \mathbb{R}_+$ is called an *admissible event times set*, if it is closed and countable, and $0 \in \mathcal{E}$. E.g. $\mathcal{E} = \{\tau_0, \tau_1, \tau_2, \ldots\}$.

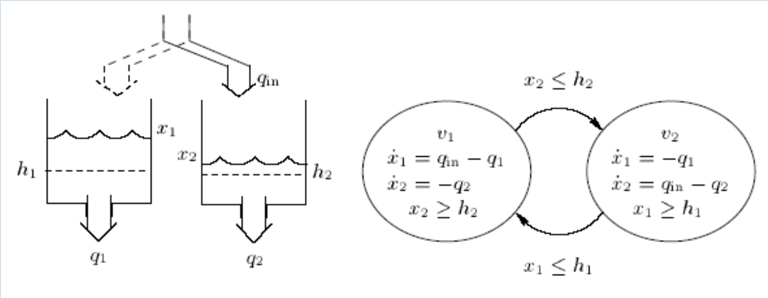
- An element t of a set $\mathcal{E}$ is said to be a *left accumulation point* of $\mathcal{E}$, if for all $t' > t$ $(t, t') \cap \mathcal{E}$ is not empty.
- It is called a *right accumulation point*, if for all $t' < t$ $(t', t) \cap \mathcal{E}$ is not empty

Definition 5 An admissible event times set $\mathcal{E}$ (or the corresponding solution) is said to be *left (right) Zeno free*, if it does not contain any left (right) accumulation points.

- Bouncing ball $\longrightarrow$ right accumulation point ...
- Time-reversed bouncing ball:

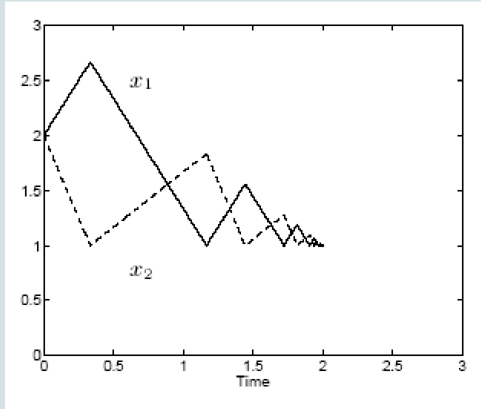


Two-tank system and Zeno behavior



A simulation

$h_1 = h_2 = 1$, $q_1 = 2$, $q_2 = 3$, $q_{\text{in}} = 4$, $x_1(0) = x_2(0) = 2$, $q(0) = v_1$



Two-tank system and Zeno behavior

- Assume total outflow $q_1 + q_2 > q_{in}$
- Control objective cannot be met and tanks will be empty in finite time
- Infinitely many switchings in finite time (right accumulation point) $\rightarrow$ right Zeno behavior

Using a non-Zeno solution concept: analysis will show that tanks do not get empty! Analysis depends crucially on solution concept!

Hybrid automaton

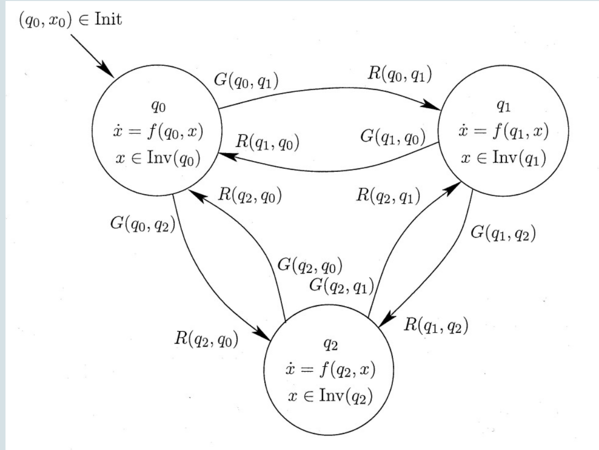
Hybrid automaton H is collection $H = (Q, X, f, \text{Init}, \text{Inv}, E, G, R)$ with

- $Q = \{q_1, \ldots, q_N\}$ is finite set of discrete states or *modes*
- $X = \mathbb{R}^n$ is set of continuous states
- $f : Q \times X \rightarrow X$ is vector field
- $\text{Init} \subseteq Q \times X$ is set of initial states
- $\text{Inv} : Q \rightarrow P(X)$ describes the *invariants*
- $E \subseteq Q \times Q$ is set of edges or *transitions*
- $G : E \rightarrow P(X)$ is *guard condition*
- $R : E \rightarrow P(X \times X)$ is *reset map*

What is what?

Hybrid automaton $H = (Q, X, f, \mathrm{Init}, \mathrm{Inv}, E, G, R)$

- Hybrid state: (q, x)
- Evolution of continuous state in mode q: $\dot{x} = f(q, x)$
- Invariant Inv: describes conditions that continuous state has to satisfy at given mode
- Guard G: specifies subset of state space where certain transition is enabled
- Reset map R: specifies how new continuous states are related to previous continuous states



Evolution of hybrid automaton

- Initial hybrid state $(q_0, x_0) \in \mathrm{Init}$
- Continuous state x evolves according to

$$\dot{x} = f(q_0, x) \quad \text{with } x(0) = x_0$$

  discrete state q remains constant: $q(t) = q_0$
- Continuous evolution can go on as long as $x \in \mathrm{Inv}(q_0)$
- If at some point state x reaches guard $G(q_0, q_1)$, then
  - transition $q_0 \longrightarrow q_1$ is enabled
  - discrete state *may* change to q_1, continuous state then jumps from current value x^- to new value x^+ with $(x^-, x^+) \in R(q_0, q_1)$
- Next, continuous evolution resumes and whole process is repeated

Hybrid time trajectory

Definition 6 A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite ($N < \infty$) or infinite ($N = \infty$) sequence of intervals of the real line, such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leq \tau_i' = \tau_{i+1}$ for $0 \leq i < N$;
- if $N < \infty$, either $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$ with $\tau_N \leq \tau_N' \leq \infty$.

- For instance,

$$\tau = \{[0,2], [2,3], \{3\}, \{3\}, [3,4.5], \{4.5\}, [4.5,6]\}$$

$$\tau = \{[0,2], [2,3], [3,4.5], \{4.5\}, [4.5,6], [6,\infty)\}$$

$$I_i = [1 - (\frac{1}{2})^i, 1 - (\frac{1}{2})^{i+1}]$$

- $\mathcal{E} = \{\tau_0, \tau_1, \tau_2, \ldots\}$
- No left-accumulations of event times ...

Execution of hybrid automaton

Definition 7 An execution χ of a HA consists of $\chi = (\tau, q, x)$

- τ a hybrid time trajectory;
- $q = \{q_i\}_{i=0}^N$ with $q_i : I_i \to Q$; and
- $x = \{x_i\}_{i=0}^N$ with $x_i : I_i \to X$

Initial condition $(q(\tau_0), x(\tau_0)) \in \mathrm{Init}$;

Continuous evolution for all i

- q_i is constant, i.e. $q_i(t) = q_i(\tau_i)$ for all $t \in I_i$;
- x_i is solution to $\dot{x}(t) = f(q_i(t), x(t))$ on I_i with initial condition $x_i(\tau_i)$ at τ_i;
- for all $t \in [\tau_i, \tau_i')$ it holds that $x_i(t) \in \mathrm{Inv}(q_i(t))$.

Discrete evolution for all i,

- $e = (q_i(\tau_i'), q_{i+1}(\tau_{i+1})) \in E$,
- $x(\tau_i') \in G(e)$;
- $(x_i(\tau_i'), x_{i+1}(\tau_{i+1})) \in R(e)$.

Well-posedness for hybrid automata

- $\mathcal{H}^{\infty}_{(q_0,x_0)}$: infinite executions: τ is an infinite sequence or if $\sum_i(\tau_i' - \tau_i) = \infty$
- $\mathcal{H}^{M}_{(q_0,x_0)}$: maximal executions: τ is not a strict prefix of another one!

- A hybrid automaton is called *non-blocking*, if $\mathcal{H}^{\infty}_{(q_0,x_0)}$ is non-empty for all $(q_0, x_0) \in \mathrm{Init}$.
- It is called *deterministic*, if $\mathcal{H}^{M}_{(q_0,x_0)}$ contains at most one element for all $(q_0, x_0) \in \mathrm{Init}$.

Well-posedness for hybrid automata - continued

Assumption

- The vector field $f(q, \cdot)$ is globally Lipschitz continuous for all $q \in Q$.
- The edge $e = (q, q')$ is contained in E if and only if $G(e) \neq \emptyset$ and $x \in G(e)$ if and only if there is an $x' \in X$ such that $(x, x') \in R(e)$.

A state $(\hat{q}, \hat{x}) \in \mathrm{Reach}$, if there exists a finite execution (τ, q, x) with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ and $(q(\tau_N'), x(\tau_N')) = (\hat{q}, \hat{x})$.

The set of states from which continuous evolution is impossible :

$$\mathrm{Out} = \{(q_0, x_0) \in Q \times X \mid \forall \varepsilon > 0 \exists t \in [0, \varepsilon)\ x_{q_0,x_0}(t) \notin \mathrm{Inv}(q_0)\}$$

in which $x_{q_0,x_0}(\cdot)$ denotes the unique solution to $\dot{x} = f(q_0, x)$ with $x(0) = x_0$.

Well-posedness theorems

Theorem A hybrid automaton is non-blocking, if for all $(q, x) \in \mathrm{Reach} \cap \mathrm{Out}$, there exists $e = (q, q') \in E$ with $x \in G(e)$. In case the automaton is deterministic, this condition is also necessary.

Theorem A hybrid automaton is deterministic, if and only if for all $(q, x) \in \mathrm{Reach}$

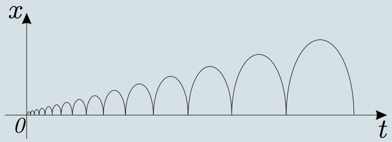
- if $x \in G((q, q'))$ for some $(q, q') \in E$, then $(q, x) \in \mathrm{Out}$;
- if $(q, q') \in E$ and $(q, q'') \in E$ with $q' \neq q''$, then $x \notin G((q, q')) \cap G((q, q''))$; and
- if $(q, q') \in E$ and $x \in G((q, q'))$, then there is at most one $x' \in X$ with $(x, x') \in R((q, q'))$.

$\longrightarrow$ no explicit / algebraic conditions and not easily verifiable $\rightarrow$ can we do more (like for DDE)?

Well-posedness issues

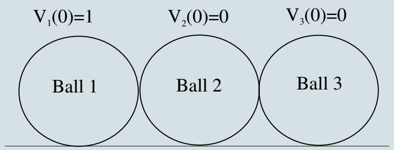
- **Initial well-posedness:** non-blocking + deterministic, i.e. absence of
  - **dead-lock**: no smooth continuation and no jump
  - splitting of trajectories
- However, no statements by previous HA theory on existence beyond
  - **live-lock**: an infinite number of jumps at one time instant, no solution on $[0, \varepsilon)$ for some $\varepsilon > 0$.
  - **right-accumulations** of event times to prevent global existence.

  or absence of
  - **left-accumulations** of event times preventing uniqueness:



Obstruction local existence

$\longrightarrow$ **Live-lock:** Infinitely many jumps at one time instant



$$v_1: \; 1 \;\; \tfrac{1}{2} \;\; \tfrac{1}{2} \;\; \tfrac{3}{8} \;\; \tfrac{3}{8} \;\; \tfrac{11}{32} \ldots \tfrac{1}{3}$$
$$v_2: \; 0 \;\; \tfrac{1}{2} \;\; \tfrac{1}{4} \;\; \tfrac{3}{8} \;\; \tfrac{5}{16} \;\; \tfrac{11}{32} \ldots \tfrac{1}{3}$$
$$v_3: \; 0 \;\; 0 \;\; \tfrac{1}{4} \;\; \tfrac{1}{4} \;\; \tfrac{5}{16} \;\; \tfrac{5}{16} \ldots \tfrac{1}{3}$$

- smooth continuation possible with constant velocity after an infinite number of events

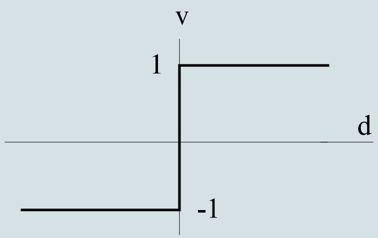
$\longrightarrow$ Exclude live-lock or show convergence of state x for local existence

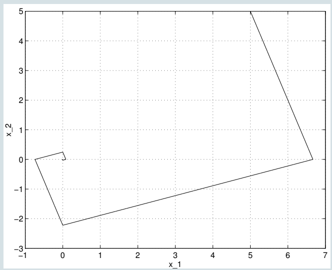
- **Discrete mode is a function of continuous state!** not for general HA!!!

Obstruction global existence: Zenoness

$\longrightarrow$ A right-accumulation of event times

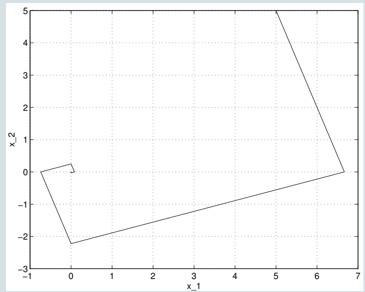
$$\begin{aligned} \dot{x}_1 &= -\operatorname{sgn}(x_1) + 2\operatorname{sgn}(x_2) \\ \dot{x}_2 &= -2\operatorname{sgn}(x_1) - \operatorname{sgn}(x_2) \end{aligned}$$





- Exclude right-accumulations or show the existence of the left-limit $\lim_{t \uparrow \tau^*} x(t)$ for global existence.
- **Discrete mode is a function of continuous state!** not for general HA!!!

Obstructions local uniqueness: Filippov's example



$$\begin{aligned} \dot{x}_1 &= \operatorname{sgn}(x_1) - 2\operatorname{sgn}(x_2) \\ \dot{x}_2 &= 2\operatorname{sgn}(x_1) + \operatorname{sgn}(x_2), \end{aligned}$$

Left accumulation point ... $\mathcal{E}$ is not left Zeno free!

Well-posedness:

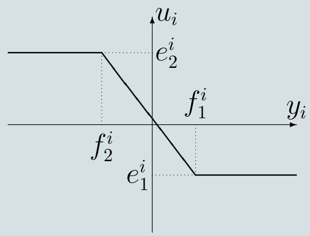
- Due to left-accumulations non-uniqueness in origin
- Using HA framework: non-blocking and deterministic
- Using Filippov's solution: non-uniqueness!

Well-posedness

- **Initially solvable** from each initial state there exists a state jump or a continuous hybrid solution on $[0, \varepsilon)$ (non-blocking)
- **Initially unique** from each initial state the jump/hybrid solution is unique (deterministic)
- **Local well-posedness** from each initial state there exists an $\varepsilon > 0$ and a hybrid solution on $[0, \varepsilon)$.
- **Global well-posedness** ... on $[0, \infty)$.

Piecewise linear systems

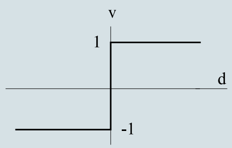
$$\mathrm{SAT}(A, B, C, D) \qquad \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ (u(t), y(t)) \in \text{saturation}_i \end{array} \qquad e_2^i - e_1^i > 0 \text{ and } f_1^i \geq f_2^i$$



Note that if $f_2^i = f_1^i$, then relay-type of nonlinearity.

Example of linear relay system: non-uniqueness

$$\begin{aligned} \dot{x} &= x - u \\ y &= x \\ u &\in -\mathrm{sgn}(y) \end{aligned}$$

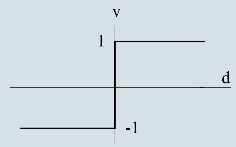


$x(0) = 0$:

- $x(t) = e^t - 1$, $(y(t) = x(t) \geq 0)$
- $x(t) = -e^t + 1$, $(y(t) = x(t) \leq 0)$
- $x(t) = 0$, $(y(t) = x(t) = 0)$

Example of linear relay system: uniqueness

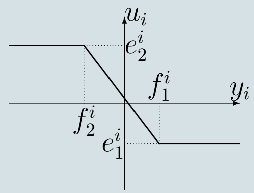
$$\begin{aligned} \dot{x} &= x + u \\ y &= x \\ u &\in -\mathrm{sgn}(y) \end{aligned}$$



$x(0) = 0$:

- $x(t) = 0$, $(y(t) = x(t) = 0)$

Piecewise linear systems



Consider SAT(A, B, C, D).

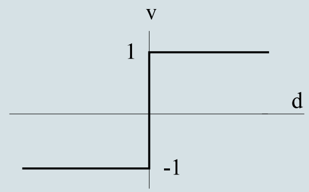
- Let R and S be the diagonal matrices with $e_2^i - e_1^i$ and $f_2^i - f_1^i$, resp.
- $G(s) = C(sI - A)^{-1}B + D$

Suppose that $G(\sigma)R - S$ is a P-matrix for all sufficiently large σ. Then, there exists a unique (left Zeno free) hybrid execution of SAT(A, B, C, D) for all initial states.

- $M \in \mathbb{R}^{m\times m}$ is a *P-matrix*, if $\det M_{II} > 0$ for all $I \subseteq \{1, \ldots, m\}$.

Linear relay systems and Filippov's solutions: left accumulations

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t)\\
y(t) &= Cx(t)\\
u(t) &\in -\mathrm{sgn}(y(t))
\end{aligned}
$$



Previous result: If $G(\sigma) = CB\sigma^{-1} + CAB\sigma^{-2} + \ldots > 0$ for sufficiently large σ, then existence and uniqueness of (left-Zeno free) executions.

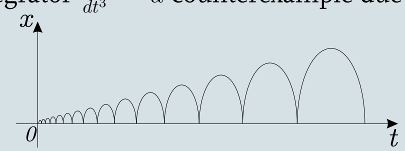
Other solution concept ...?

Filippov's solutions include left-accumulations and satisfy $\dot{x} \in F(x)$ almost everywhere, with

- $F(x) = \{Ax + B\}$ for $Cx < 0$
- $F(x) = \{Ax - B\}$ for $Cx > 0$
- $F(x) = \{Ax + B\bar{u} \mid \bar{u} \in [-1, 1]\}$ when $Cx = 0$

In case of relative degree 1 ($CB > 0$) and relative degree 2 ($CAB > 0$) sufficient for Filippov uniqueness.

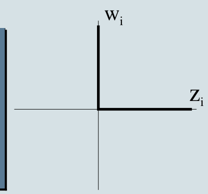
However, triple integrator $\frac{d^3x}{dt^3} = u$ counterexample due to:



So, (other) example of HA uniqueness (deterministic), but non-uniqueness in "Filippov"

Linear complementarity systems

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bz(t)\\
w(t) &= Cx(t) + Dz(t)\\
0 \leq w(t) &\perp z(t) \geq 0
\end{aligned}
$$

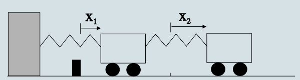


$$\{z_i(t) = 0 \text{ and } w_i(t) \geq 0\} \text{ or } \{w_i(t) = 0 \text{ and } z_i(t) \geq 0\}$$

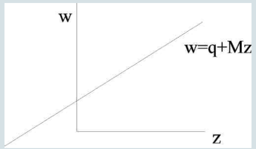
- Modes parameterized by $I \subseteq \{1, \ldots, k\}$ such that

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bz(t)\\
w(t) &= Cx(t) + Dz(t)\\
w_i(t) &= 0,\ i \in I \text{ and } z_i(t) = 0, i \notin I
\end{aligned}
$$

- Resets!



Example 1

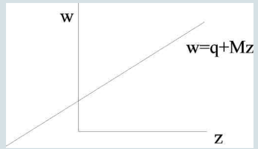


$$\begin{aligned} \dot{x} &= x + z \\ w &= x - z \\ 0 &\leq w \perp z \geq 0 \end{aligned}$$

- $z = 0$: $\dot{x} = x$, $w = x \geq 0$
- $w = 0$: $\dot{x} = 2x$, $z = x \geq 0$

Hence, $x(0) = 1$ two solutions and $x(0) = -1$ no solution trajectory!

Example 2



$$\begin{aligned} \dot{x} &= x + z \\ w &= x + z \\ 0 &\leq w \perp z \geq 0 \end{aligned}$$

- $z = 0$: $\dot{x} = x$, $w = x \geq 0$
- $w = 0$: $\dot{x} = 0$, $z = -x \geq 0$

Existence and uniqueness!

Well-posedness including jumps

- **Initially solvable** from each initial state there exists a state jump or a continuous hybrid solution on $[0, \varepsilon)$ (non-blocking)
- **Initially unique** from each initial state the jump/hybrid solution is unique (deterministic)
- **Local well-posedness** from each initial state there exists an $\varepsilon > 0$ and a hybrid solution on $[0, \varepsilon)$.
- **Global well-posedness** ... on $[0, \infty)$.

Local well-posedness (including jumps)

$\dot{x}(t) = Ax(t) + Bz(t), \quad w(t) = Cx(t) + Dz(t), \quad 0 \leq z(t) \perp w(t) \geq 0$

Markov parameters: $H^0 = D$ and $H^i = CA^{i-1}B$, $i = 1, 2, \ldots$

$$\eta_j = \inf\{i \mid H^i_{\bullet j} \neq 0\}, \; \rho_j = \inf\{i \mid H^i_{j\bullet} \neq 0\},$$

The *leading row and column coefficient matrices* $\mathcal{M}$ and $\mathcal{N}$

$$\mathcal{M} := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \text{ and } \mathcal{N} := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k})$$

- $M \in \mathbb{R}^{m \times m}$ is a *P-matrix*, if $\det M_{II} > 0$ for all $I \subseteq \{1, \ldots, m\}$.

Local well-posedness (including jumps)

$$\dot{x}(t) = Ax(t) + Bz(t), \quad w(t) = Cx(t) + Dz(t), \;\; 0 \leq z(t) \perp w(t) \geq 0$$

Markov parameters: $H^0 = D$ and $H^i = CA^{i-1}B$, $i = 1, 2, \ldots$

$$\eta_j = \inf\{i \mid H^i_{\bullet j} \neq 0\}, \;\; \rho_j = \inf\{i \mid H^i_{j\bullet} \neq 0\},$$

The *leading row and column coefficient matrices* $\mathcal{M}$ and $\mathcal{N}$

$$\mathcal{M} := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \quad \text{and } \mathcal{N} := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k})$$

If $\mathcal{N}$ and $\mathcal{M}$ are P-matrices, then LCS(A, B, C, D) has for all x_0 a unique left Zeno free execution on an interval of the form $[0, \varepsilon)$ for some $\varepsilon > 0$.

- Moreover, live-lock does not occur: at most one jump
- Necessary and sufficient for **global** well-posedness for **bimodal** LCS

Summary

- Smooth differential equations
  - Solution concept straightforward
  - Lipschitz continuity sufficient for well-posedness
  - absence Lipschitz: possibly non-uniqueness
  - absence global Lipschitz: possibly finite escape times and no global existence
- Switched systems (discontinuous differential equations)
  - Sliding modes (Filippov's convex or Utkin's equivalent control definition)
  - Solution concept from differential inclusions
  - (Local) existence of solutions guaranteed.
  - Well-posedness: directions of vector field at switching plane

"No events"

Summary - continued

- Hybrid systems:
  - Complications due to Zeno
  - Relation between solution concept and well-posedness and analysis
    - Tanks stay full along non-Zeno solutions!!!
    - Filippov's example has unique non-Zeno solutions, but non-unique Zeno solutions
  - Well-posedness
    - Initial well-posedness (non-blocking and deterministic)
    - Local well-posedness: $[0, \varepsilon)$ (live-lock)
    - Global well-posedness: $[0, \infty)$ (right-accumulations)
  - Conditions for hybrid automata: implicit!
  - Algebraic conditions for certain classes with more structure!

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Thanks for attention!

Questions?