A quick tour of Piecewise-smooth and Complementarity Systems: Analysis, Numerics and Bifurcations

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Outline

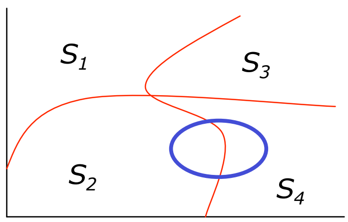
- Modelling Nonsmooth Systems:
  - PWS ODEs
  - Hybrid Systems
  - Complementarity Systems
- Solution concepts and well-posedness
- Structural Stability and bifurcations
- Numerical analysis:
  - Simulation (time-stepping, event-driven)
  - Continuation
  - The SICONOS platform
- Applications to Power Electronics, Mechanics and Control
- If you want to find out more...

Nonsmooth Systems

- We are interested in studying a class of systems whose vector field is nonsmooth
- There are several frameworks to describe such systems:
  - Nonsmooth sets of ODEs
  - Complementarity Systems
  - Differential Inclusions
  - Measure differential inclusions
  - Hybrid Dynamical Systems
- Let's look at three of these formalisms…

1. Piecewise-Smooth ODEs

$$\dot{x} = \begin{cases} F_1(x,\mu), & x \in S_1, \\ F_2(x,\mu), & x \in S_2, \\ F_3(x,\mu), & x \in S_3, \\ \vdots & \\ F_N(x,\mu), & x \in S_N, \end{cases}$$



- The system is discontinous across the boundaries between different regions (*switching manifolds*)
- Discontinuities in the states can be accounted for by adding appropriate equations

- Consider a sufficiently small region D such that:

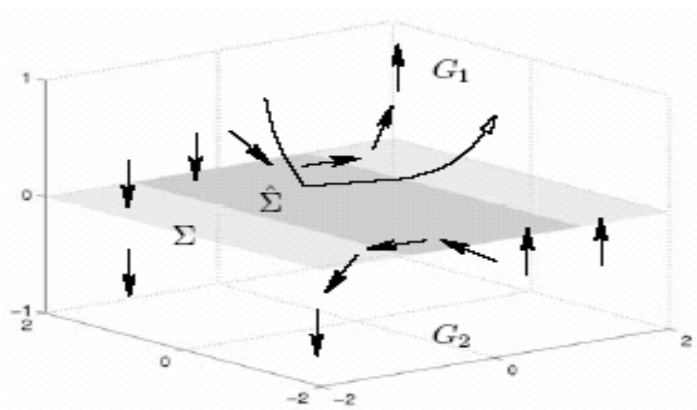
$$\dot{x} = \begin{cases} F_1(x,\mu), & H(x,\mu) > 0 \\ F_2(x,\mu), & H(x,\mu) < 0 \end{cases}$$

- H defines the boundary where smoothness is lost

$$\Sigma := \left\{ x \in R^n : H(x) = 0 \right\}$$

- D is partitioned in two regions G_1 and G_2

Phase Space Topology



Different types of NS systems

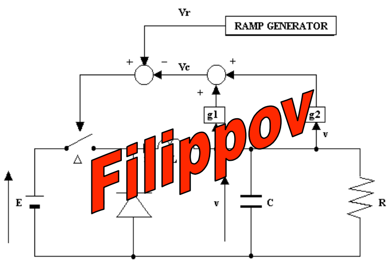
- Nonsmooth systems can be classified by their *degree of discontinuity* across the boundary
  - Systems with discontinuos state jumps (e.g. impact oscillator)
  - Systems with discontinuos vector field or *Filippov* (e.g. relay systems)

$$F_1(x) \neq F_2(x)$$

  - Piecewise Smooth Continuous systems

$$F_1(x) = F_2(x), \quad \frac{\partial F_1}{\partial x} \neq \frac{\partial F_2}{\partial x}$$

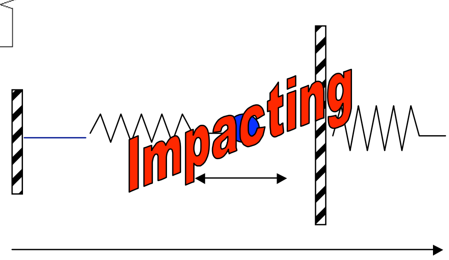
Examples

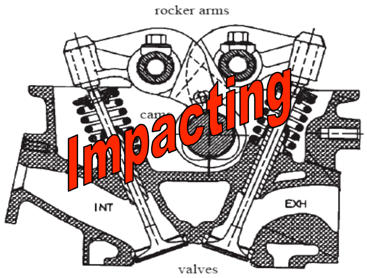


$$\begin{cases} \dfrac{di}{dt} = -\dfrac{1}{L}v(t) + \dfrac{\delta(t)}{L}E \\ \dfrac{dv}{dt} = \dfrac{1}{C}i(t) - \dfrac{1}{RC}v(t) \end{cases}$$

$$\ddot{y} + c\dot{y} + ky = \alpha\cos(\omega t),$$

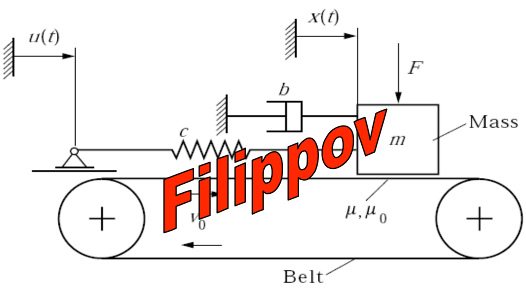
$$\dot{y} \to -r\dot{y}, \; y = \sigma$$





An overhead camshaft automotive valve train

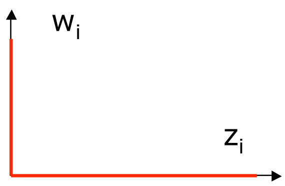
Friction Oscillators



2. Complementarity Systems

- An alternative framework can be used to describe nonsmooth systems, in particular nonsmooth mechanical systems [Brogliato, 2002]
- Complementarity systems have been studied in mechanics for a long time
- They consist of equations of the form:

$$\dot{x} = f(x(t), z(t))$$
$$w(t) = h(x(t), z(t))$$
$$0 \le w \perp z \ge 0$$



Linear Complementarity Systems

$$\dot{x} = Ax + Bz$$
$$w = Cx + Dz$$
$$0 \le w \perp z \ge 0$$

ODEs

$$\dot{x} = \begin{cases} A_1 x + b\mu, & c^T x > 0 \\ A_2 x + b\mu, & c^T x < 0 \end{cases}$$

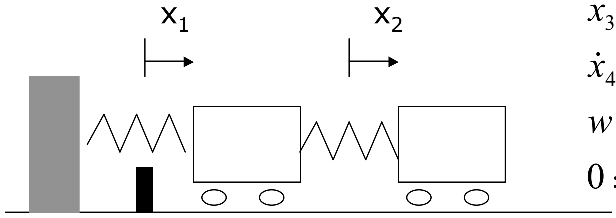
$$A_1 - A_2 = dc^T$$

LCS

$$\begin{cases} \dot{x} = A_2 x + b\mu + d\lambda \\ w = -c^T x + \lambda \\ 0 \le w \perp \lambda \ge 0 \end{cases}$$

- Complementarity systems are particularly suited to describe systems with unilateral constraints (diodes, impact oscisllators, friction, saturations, relays, VSS)
- Routines from optimization can be used to solve the LCP
- The formalism is compact while retaining its physical meaning…

Example



$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = -2x_1 + x_2 + z$$
$$\dot{x}_4 = x_1 - x_2$$
$$w = x_1$$
$$0 \leq w \perp z \geq 0$$

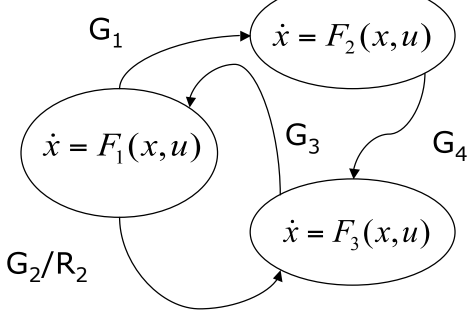
- Note that at the impact $w = 0$, hence $x_1 = 0 = x_3$ and $z > 0$ represents the reaction force !

- More compact than sets of ODEs

3. Hybrid Dynamical Systems

- Another very general framework has been introduced in control theory
- It helps describing systems whose dynamics are hybrid: both continuous-time and discrete-time (e.g. digital control etc)
- Here the system consists of a sextuple
  $H = (Q, E, D, F, G, R)$

  Q : set of discrete states
  E: collection of the edges
  D: domains of H
  F: collection of vector fields
  G: collection of guards
  R: collection of resets



- Hybrid Systems are a very general framework, encompassing a wide range of cases

- They can be seen as PWS systems of the form:

$$\dot{x} = f_{q(h)}(x,u)$$
$$q(h+1) = M(q(h))$$

- They can be useful but usually too general leading to a very cumbersome formalisation

To recap

- We have seen sofar three alternative frameworks that can be used to model nonsmooth dynamical systems
- Each has its pros and cons
- We will refer mostly to PWS sets of ODEs and complementarity
- In all of these cases the first problem that needs to be address is the well-posedness of the system solutions

Solution concepts

- Several attempts have been made to define the concept of solution for nonsmooth systems
- There are many theoretical problems connected with the well posedness, reversibility etc.
- To name just a few:
  - *Chattering or Zeno phenomena*
  - *Sliding (or Filippov) solutions*
  - *Uniqueness*
- We will give a brief outline of the main concepts

Well posedness: a simple example

- Take the system:

$$\dot{x} = x + z$$
$$w = x - z$$
$$0 \le w \perp z \ge 0$$

- Then:

$$z = 0 \Rightarrow \dot{x} = x, \quad w = x \ge 0$$
$$w = 0 \Rightarrow \dot{x} = 2x, \quad z = x \ge 0$$

- If $x(0) = 1$ we have 2 solns, while if $x(0) = -1$ no solution !

Well posedness: another example

- Now, take the system:

$$\dot{x} = x + z$$
$$w = x + z$$
$$0 \le w \perp z \ge 0$$

- Then:

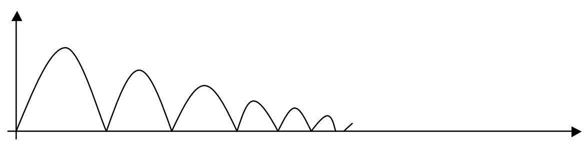
$$z = 0 \Rightarrow \dot{x} = x, \quad w = x \ge 0$$
$$w = 0 \Rightarrow \dot{x} = 0, \quad z = -x \ge 0$$

- Now we have existence and uniqueness !!!

Other problems

- Chattering or **Zeno phenomena**

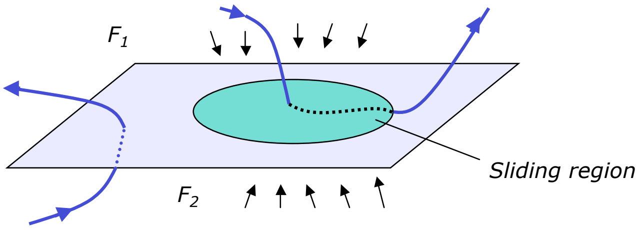
Accumulation of infinite impacts in finite time causing deadlock in numerical simulations (sometimes loss of uniqueness)

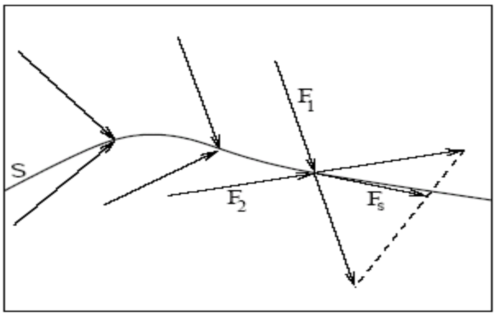


- In Filippov systems presence of **sliding solutions**, i.e. solutions lying on the discontinuity set

Sliding mode in Filippov Systems

- If $<\nabla H, F1> < \nabla H, F2> < 0$ we can have sliding modes, i.e. solutions constrained on Σ
- There can be regions $\hat{\Sigma} \subset \Sigma$ where sliding is possible (sliding regions)
- What happens then if, by varying the parameters, the system trajectory hits the sliding region ?





- Sliding can be studied by using Filippov convex method. Basically we find a vector field which lies in the convex hull of F_1 and F_2 and is tangential to the switching manifold

$$F_s = \frac{F_1 + F_2}{2} + H_u \frac{F_2 - F_1}{2},$$

- Since we want the sliding vector field to be tangential to the switching manifold, we have:

$$\langle \nabla H, F_s \rangle = 0.$$

- Hence we can write:

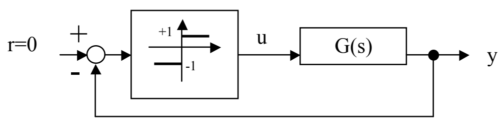
$$H_u(x) = -\frac{\langle \nabla H, F_1 \rangle + \langle \nabla H, F_2 \rangle}{\langle \nabla H, F_2 \rangle - \langle \nabla H, F_1 \rangle}.$$

- And the swtiching manifold can be given as:

$$\hat{\Sigma} := \{x \in \Sigma : |H_u(x)| < 1\},$$

Relay Control System

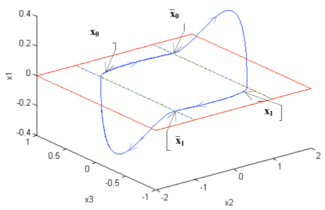
- A classical example of system with sliding are relay control systems



$$\dot{x} = Ax + bu, \quad y = c^T x, \quad u = -sign(y)$$

- Using Filippov method we can characterise these systems (see board…)

- Note that in general sliding segment can become part of periodic solutions of the system under investigation (important for bifurcation analysis)



- ✓ *stick-slip oscillations*
- ✓ *chattering orbits*
- ✓ *complex relay dynamics* [Kowalczyk, di Bernardo 01]
- ✓ friction oscillators, power converters, vibro-impacting devices

To recap

- Well posedness of solutions of nonsmooth systems is still an open challenge
- Some tools are available to characterise, for example, sliding solutions in Filippov systems
- A complete theory is not available. Maybe a general theory would be too complicated to be useful !

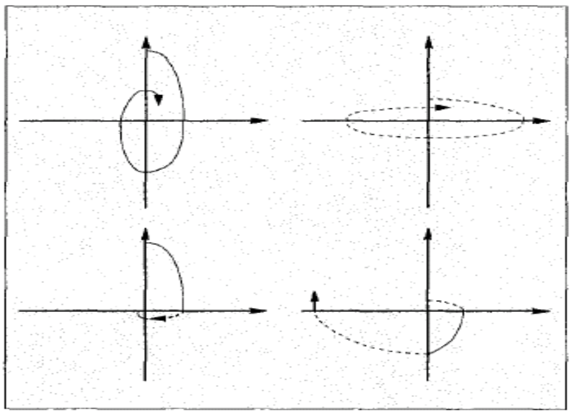
- An even more striking problem is stability !

Stability

- Switching between two or more vector fields can make stable systems unstable or viceversa !
- Namely examples can be found where a switched system is unstable even if all individual systems are stable
- The viceversa is also true !
- So the problem of studying stability becomes:

  i. *Find conditions that guarantee stability of a switched system for arbitrary switching signals*
  ii. *Find constraints on switching signals that guarantee stability*

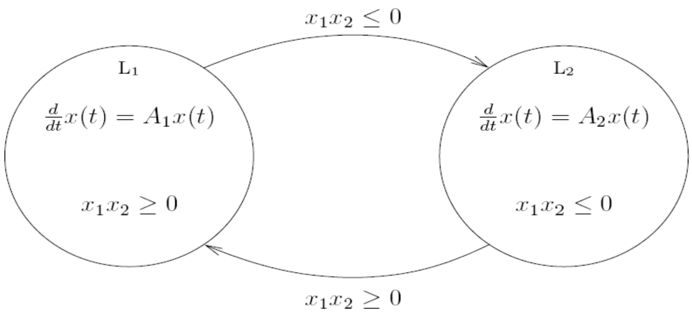
Possible trajectories

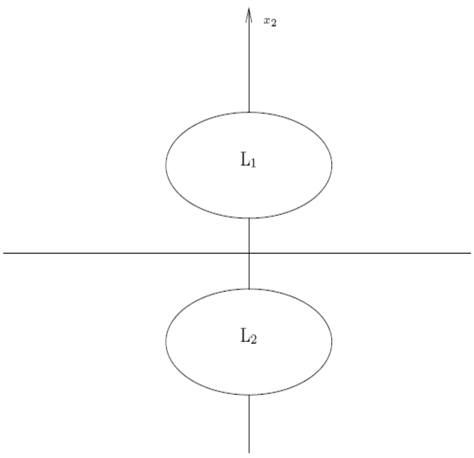


Example

EXAMPLE

$$A_1 = \begin{bmatrix} -0.01 & -0.5 \\ 2 & -0.01 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -0.01 & -2 \\ 0.5 & -0.01 \end{bmatrix}$$





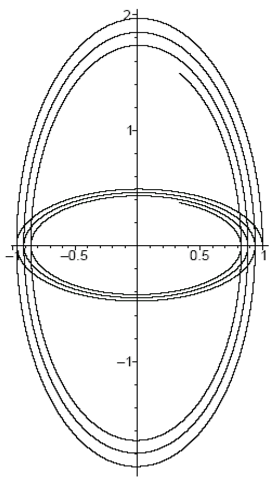
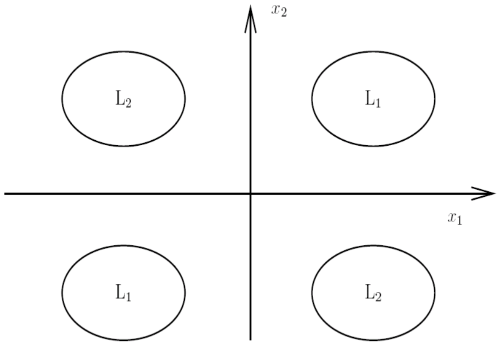
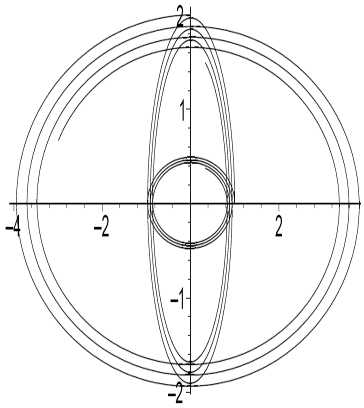


Figure 2: Phase portrait





- So, the stability properties depend not only on the dynamics of the system in each region but also on the switching policy between subsystems !
- There are many attempts to find sufficient conditions for stability (*Hot research topic !!!)*
- Very few available !

Common Lyapunov functions

- One of the most quoted results is that a switched system is A.S. if one can find a so-called common Lyapunov function
- Namely *if all systems share a radially unbounded common Lyapunov fucntion than the switched system is globally asymptotically stable*

- Note that the reverse is not true
- CLF difficult, at times impossible, to find
- Result might be too general so we need some more practical solutions

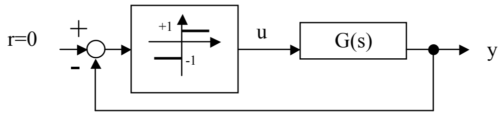
Poincare' maps

- Generally to study the stability of some solutions (e.g. periodic solution) we can use Poincare' maps
- Two types of maps can be defined for nonsmooth systems in general:
  - Stroboscopic maps (uniform sampling)
  - Impact or Switching maps (non-uniform sampling)
- Despite carrying the same amount of information, these two maps can offer different perspectives

- Typically these maps can be obtained numerically but at times they can be also obtained analytically
- In general we get maps in an implicit form (see board)
- They can be used to study existence and stability of solutions

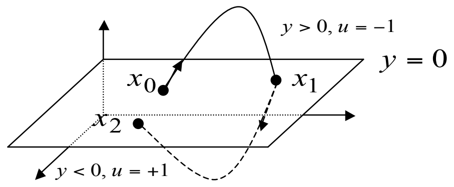
Relay Control System

- A classical example of system with sliding are relay control systems



$$\dot{x} = Ax + bu, \quad y = c^T x, \quad u = -sign(y)$$

- The switching map allow the computation of the existence and stability of periodic solutions



$$x_1 = \Pi^+(x_0, \delta_{01}) = N(\delta_{01})x_0 - M(\delta_{01}),$$

$$x_2 = \Pi^-(x_1, \delta_{12}) = N(\delta_{12})x_1 + M(\delta_{12}),$$

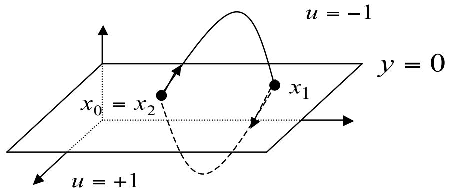
$$N(\delta) = e^{A\delta}, \quad M(\delta) = A^{-1}[N(\delta) - I]b,$$

- switching map

$$x_2 = \Pi^-(\Pi^+(x_0, \delta_{01}), \delta_{12})), \quad \Pi = \Pi^- \circ \Pi^+$$

- switching conditions: $c^T x_1 = 0, \quad c^T x_2 = 0$

- Simple necessary conditions for the existence can be obtained:



$$x_2 = x_0$$

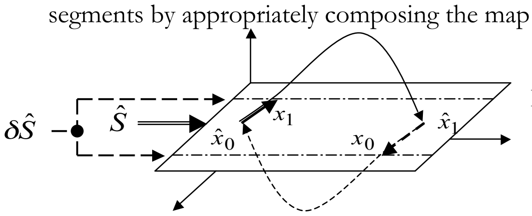
$$\delta_{01} = \delta_{12} = \delta$$

$$x_1 = -x_0$$

- Solving

$$c^T [I + N(\delta]^{-1} M(\delta) = 0$$

- We can also take into account the presence of sliding segments by appropriately composing the map



$$\Sigma : \hat{x}_0 \mapsto x_1$$
$$\Pi^+ : x_1 \mapsto \hat{x}_1$$
$$\Sigma : \hat{x}_1 \mapsto x_0$$
$$\Pi^- : x_0 \mapsto \hat{x}_0$$

- And the overall map becomes: $\Theta = \Theta^- \circ \Theta^+ = (\Sigma \circ \Pi^-) \circ (\Sigma \circ \Pi^+)$
- The sliding map can be derived from the sliding flow

$$\dot{y} = 0 \Rightarrow c^T(Ax + bu_{eq}) = 0 \Rightarrow u_{eq} = -(c^T b)^{-1} c^T Ax$$
$$\dot{x} = [I - b(c^T b)^{-1} c^T] Ax = \hat{A}x \Rightarrow \Sigma(x) = e^{\hat{A}\delta_s} x$$

Stability Analysis

- Switching maps can also be used for stability analysis… (see board)
- So to recap…
- Poincare' maps can be used to derive some conditions for existence and stability
- They are useful for both numerical and analytical purposes

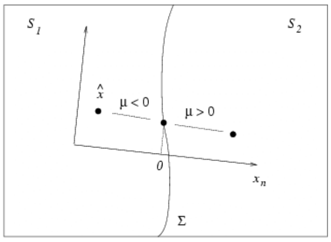
Structural Stability

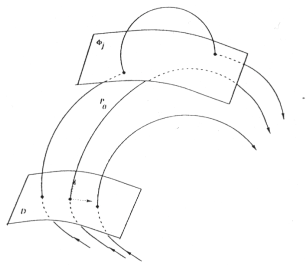
- Another important aspect is the structural stability of hybrid systems which can be better understood using complementarity of PWS models
- The problem is to study and classify mechanisms through which the system phase space looses its structural stability, e.g. *bifurcations*
- Note that currently there is no formal agreement on the concept of *topological equivalence* for nonsmooth systems
- For example, does the topology allows for a change in the number or relative positions of the discontinuity boundaries under parameter variations ? Or the degree of discontinuity across such boundaries ?

Discontinuity-induced Bifurcations

- Obviously, nonsmooth systems can exhibit standard bifurcations (e.g. Jacobian of Poincare' map can become singular)
- Here we take a pragmatic approach. We are interested in studying situations which are unique to nonsmooth systems (or *DIBs*)
- Specifically when the system dynamics does something *degenerate* w.r.t. a discontinuity boundary
- For example, an invariant set gaining a first contact with a certain Σ or the appearance of sliding along the orbits of that invariant set

- We concentrate on DIBs which involve the simplest types of invariant sets: *equilibria* or *periodic orbits*
- Nonstandard bifurcations due to the *interaction between trajectories (Ω-limit set) and discontinuity sets*
- Different scenarios according to the properties of the vector field at the *boundary*





Bifurcations in PWS systems: overview

- Standard: *SN, PD etc.*
- Discontinuity-induced *(C-bifurcations* [Feigin 70]):

- **PWS maps:** ***Border Collisions*** *of fixed points* [Nusse, Yorke 92], [Feigin 70s]
- **PWS flows:**
  - *Discontinuous bifurcations* of equilibria [Leine 03], [di Bernardo et al 04]
  - ***Grazing Bifurcations*** *of periodic orbits* [Whiston 87][Nordmark 91]
  - ***Sliding Bifurcations*** [Feigin 94][diBernardo et al 98]

Classification

- In applications, it is important to possess strategies to:
  - Detect the occurrence of C-bifurcations
  - Predict the dynamical scenario following their occurrence
- i.e. assessing the structural stability of the system under investigation (i.e. persistence of some desired behaviour etc.)

Classification Strategies

- In smooth systems, this can be done by using appropriate analytical conditions to distinguish between saddle-nodes, period-doublings, Hopf bifurcations etc.
- What about nonsmooth systems ?
- Currently there exist no general classification strategy for DIBs
- Many results are available for PWS discrete systems (maps)

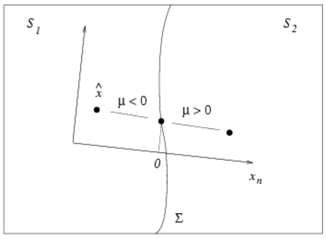
Border Collisions in PWS maps

- Consider a map of the form:

$$x_{k+1} = \begin{cases} F_1(x_k, p), & H(x_k) < 0 \\ F_2(x_k, p), & H(x_k) > 0 \end{cases}$$

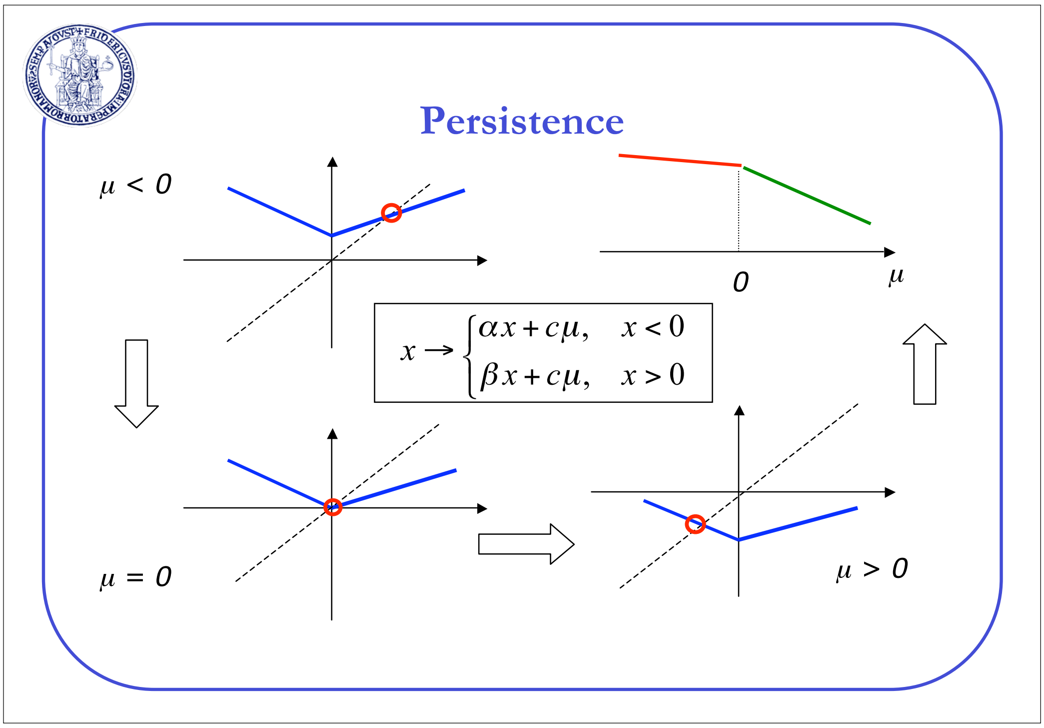
We say that a fixed point is undergoing a *border-collision* bifurcation at $p=0$ if:

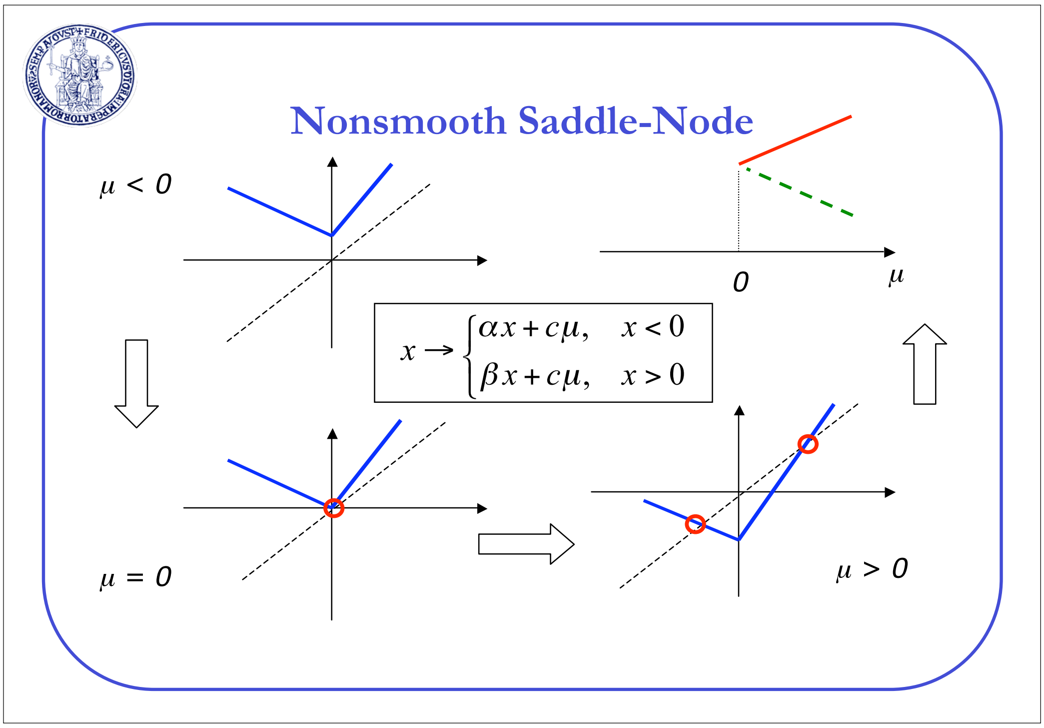
1. $\mu \in (-\varepsilon, 0) \Rightarrow x^* \in S_1$
2. $\mu \in (0, \varepsilon) \Rightarrow x^* \in S_2$
3. $\mu = 0 \Rightarrow x^* \in \Sigma$
4. $DF_1 \neq DF_2$ on Σ



Classifying Border Collisions

- When a border-collision occurs several scenarios are possible
- This can be illustrated by means of a very simple 1D map…


Persistence
μ < 0
0
μ
$x \to \begin{cases} \alpha x + c\mu, & x < 0 \\ \beta x + c\mu, & x > 0 \end{cases}$
μ = 0
μ > 0

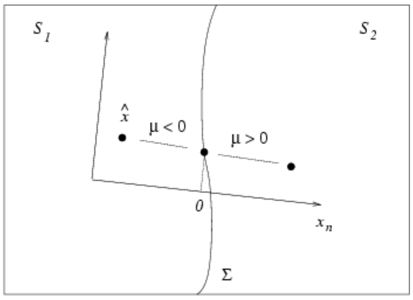

Nonsmooth Saddle-Node
μ < 0
0
μ
$x \to \begin{cases} \alpha x + c\mu, & x < 0 \\ \beta x + c\mu, & x > 0 \end{cases}$
μ = 0
μ > 0

- Thus, we can classify border-collisions by studying the properties of the map about the bifurcation point

- Linearising the map x* we then get:

$$x \to \begin{cases} A_1 x + B\mu, & Cx < 0 \\ A_2 x + B\mu, & Cx > 0 \end{cases}$$

where $A_2 = A_1 + EC$ for some vector E *(PWLC)*



Feigin's classification strategy

- We can now classify the bifurcation scenarios observed at a border collision by studying the eigenvalues of A_1 and A_2

- Namely say:

σ_1^+ : no. of eigenvalues of A_1 greater than 1

σ_2^+ : no. of eigenvalues of A_2 greater than 1

σ_1^- : no. of eigenvalues of A_1 less than –1

σ_2^- : no. of eigenvalues of A_2 less than 1

- It is possible to show that after the border-collision the orbit involved in the bifurcation will behave as follows:

1. smoothly changes into one containing an additional section in the other region of the phase space, if

$$\sigma_1^+ + \sigma_2^+ \text{ is even } (\textit{Persistence})$$

2. suddenly disappears after touching the boundary if

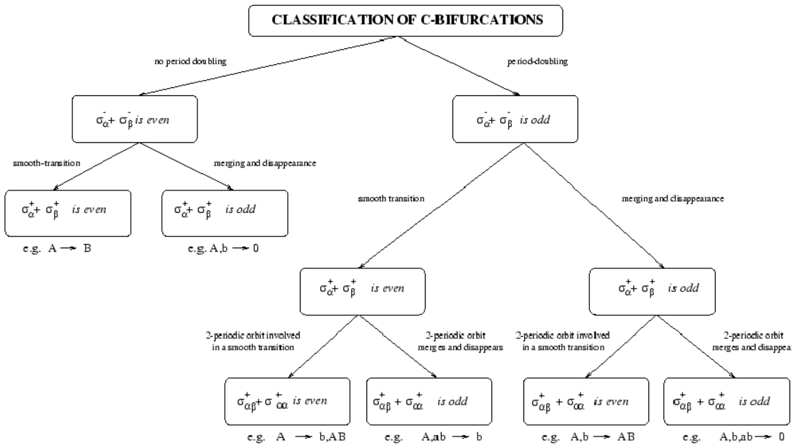
$$\sigma_1^+ + \sigma_2^+ \text{ is odd } (\textit{Nonsmooth SN})$$

3. undergoes a period-doubling, if

$$\sigma_1^- + \sigma_2^- \text{ is odd } (\textit{Nonsmooth PD})$$

An hint of the proof

- This result can be proven by relatively simple algebra… (see [diBernardo et al, 1999])
- The simple conditions that were derived can be used to derive bifurcation scenarios of increasing complexity
- Namely, using conditions for the existence of higher periodic solutions, one can construct the following *classification tree*



Remarks

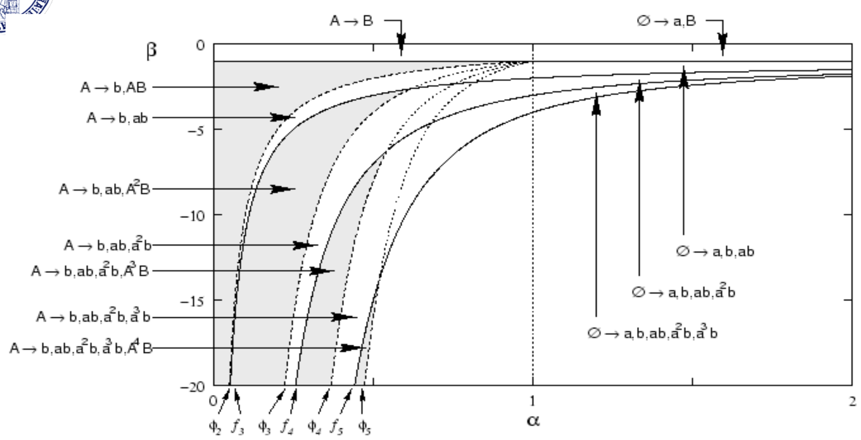
- Note that some scenarios predict the transition from a stable solution to an unstable solution or even no solution e.g. $A \rightarrow b, ab$ or $A,b \rightarrow 0$
- In these cases we should look for other possible attractors
- In *n*-dimensions this is too difficult (i.e. proving existence of chaos for example)
- A complete classification is only possible in 1D and 2D
- Let’s look at some examples

Example: One-dimensional map

- This allows the analytical classification of nonsmooth transitions of fixed points in maps
- Take for example a simple 1D map of the form

$$
\begin{aligned}
x^{(n+1)} &= \alpha x^{(n)} - \mu \qquad & x^{(n)} \geqslant 0 \\
x^{(n+1)} &= \beta x^{(n)} - \mu \qquad & x^{(n)} \leqslant 0
\end{aligned}
$$

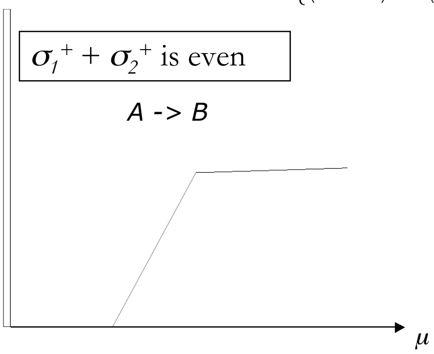
- Then it can be shown that its fixed points will undergo the following set of bifurcations

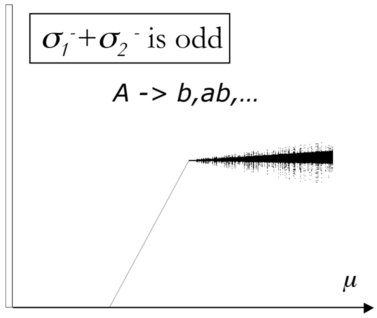


- We can prove the occurrence of chaos using Sharkovski
- A complete classification is only possible in 1D and 2D

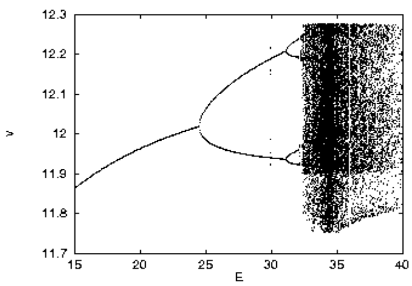
Example: 2D map

$$x \rightarrow \begin{cases} \begin{pmatrix} a_{11} & 1 \\ a_{12} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu, & x_1 < 0 \\ \begin{pmatrix} a_{21} & 1 \\ a_{22} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu, & x_1 > 0 \end{cases}$$

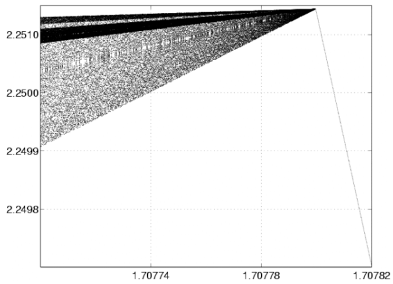




- More complex transitions are possible
- To account for some of them we must take into account the next class of DIBS, that of limit cycles



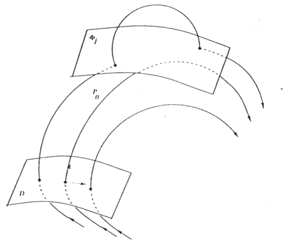
DC-DC buck converter



Friction oscillators

DIBs of limit cycles (Grazing)

- At the grazing point the trajectory hits tangentially the switching manifold

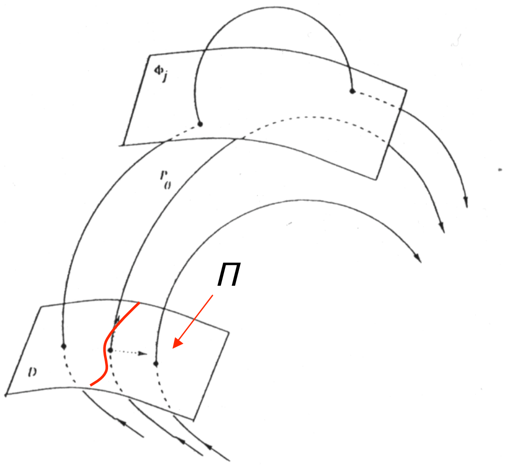


$$H(\hat{x}) = 0, \quad H_x(\hat{x}) \neq 0$$
$$\langle H_x, F_i \rangle = 0, \quad \langle H_x, F_{ix} F_i \rangle + \langle H_{xx} F_i, F_i \rangle > 0$$

- Typically, it is assumed that **Σ** is never simultaneously attracting from G_1 and G_2
- Otherwise, sliding is possible and we might have *sliding bifurcations*

Analysis and Classification

- To study this bifurcation scenario, we can associate a map to a grazing orbit
- Namely if we say *G* the locus on *Π* associated to grazing orbits…
- … we can associate a PWS map to the grazing orbit



- The key is to have the map analytically in order to be able to classify the bifurcation scenarios close to a grazing

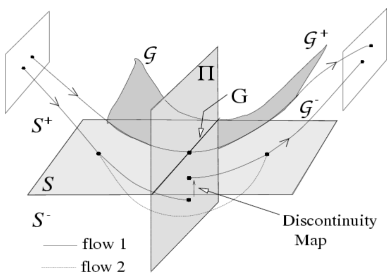
- **Important:**

  - ✓ How do we construct such mapping ?
  - ✓ Is this map always PWL as some time suggested in the literature ???

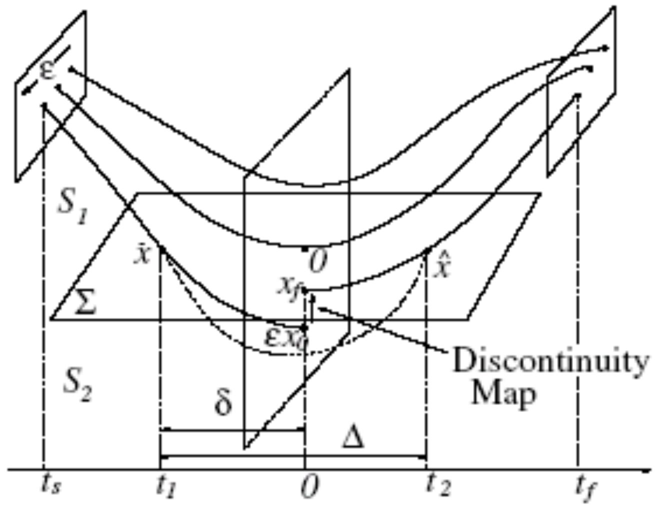
- Note that if the map is PWLC then we can classify grazing scenarios using the theory of BC bifurcations of fixed points

Discontinuity Maps

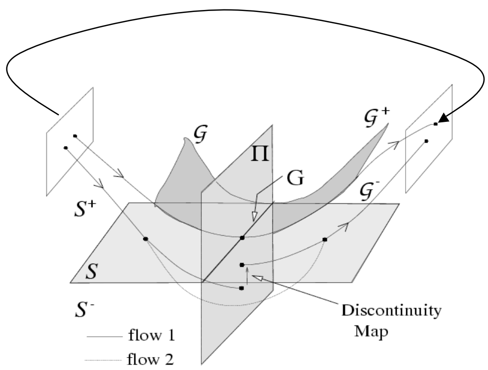
- To answer these questions we use the concept of *discontinuity mapping* [Nordmark et al. 99]



- The aim is to derive a map which gives the *correction to be made to the system trajectories in order to account for the presence of the switching manifold in phase space*



➢ The DM can be composed with the affine transformation describing the periodic part of the bifurcating orbit to obtain the global poincarè map to be used for bifurcation analysis



- The discontinuity map can be used to classify analytically the scenarios following a grazing bifurcations.
- Many other bifurcation scenarios are possible:
  - Boundary-equilibrium bifurcations
  - Corner-collisions and corner-impacts
  - Zeno-bifurcations
- To find out more see list of references at the end of the talk

Numerical Simulation of Switching Systems

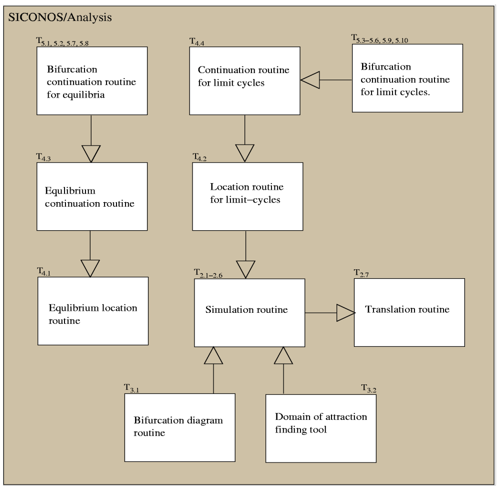
- Time-driven vs. event-driven
- Existing software not always reliable: *Stateflow*
- Typical problems: sliding and zeno
- Important: accuracy can be a problem (see board)
- Two examples:

  - Matlab/Stateflow
  - Complementarity systems

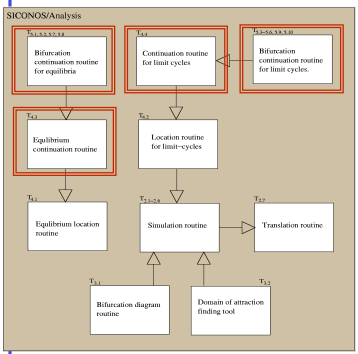
Numerical Continuation

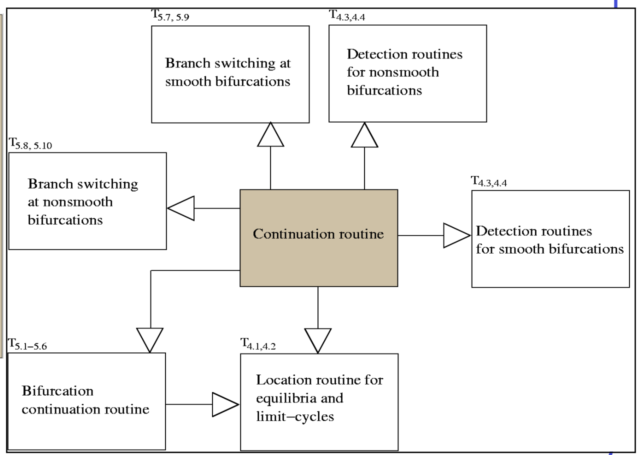
- Numerical Simulation is just a part of the story. The other important issue is Continuation
- No equivalent of AUTO or MATCONT available for nonsmooth systems
- SICONOS platform implements some brute-force and some continuation routines for:
  - Continuation of equilibria and limit cycles
  - Bifurcation detection (smooth and DIBs)
  - Regions of Stability
- Still lots of work to be done

SICONOS/Analysis

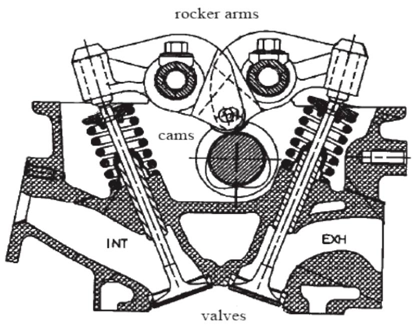


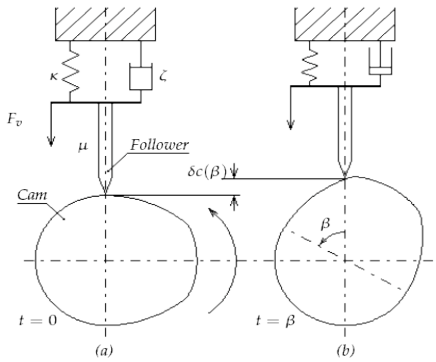
SICONOS/Analysis



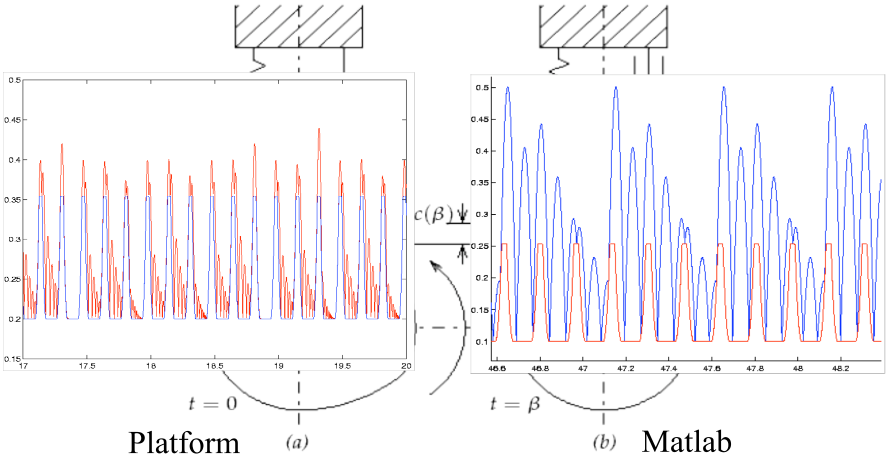


Examples – Cam

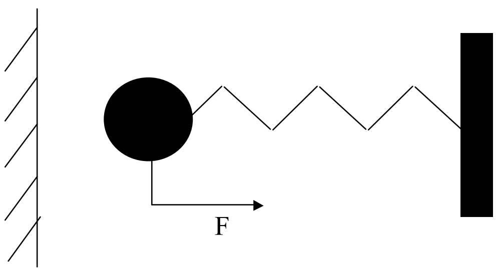




Examples – Cam

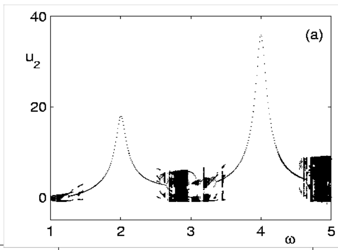


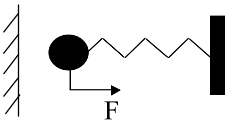
Examples – Impact oscillator

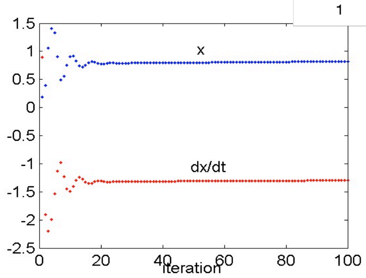


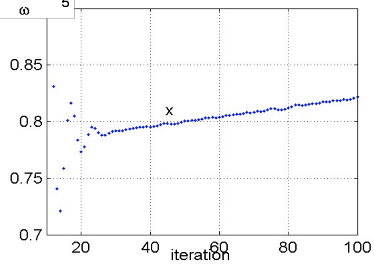
Examples – Attractor

Impact Oscillator



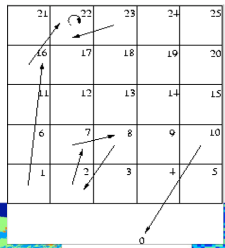


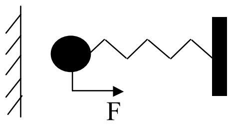




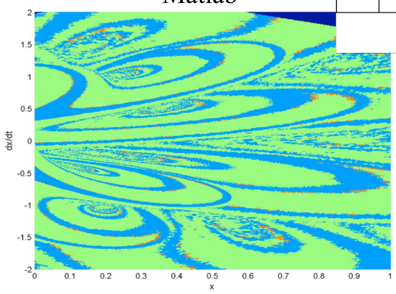
Examples – Domain of Attraction

Impact Oscillator

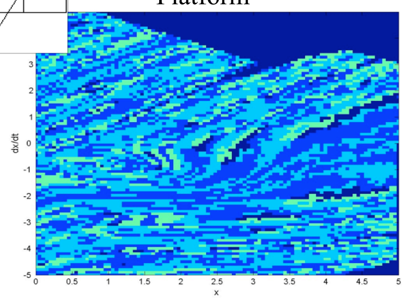




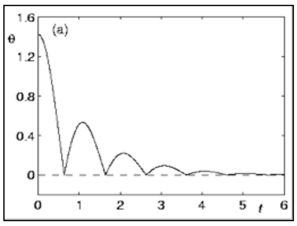
Matlab

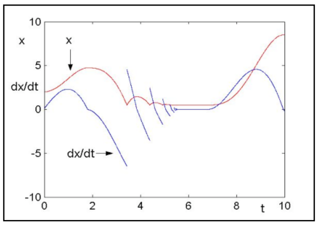


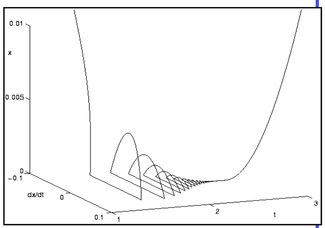
Platform

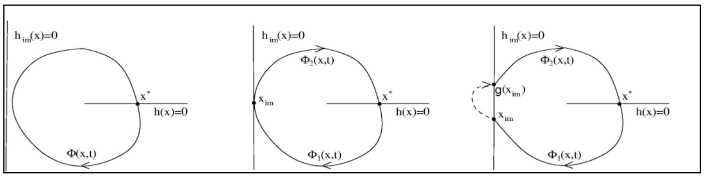


Grazing and Chattering







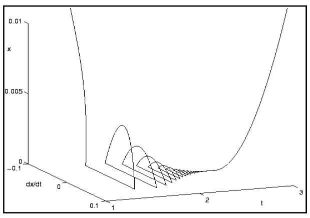


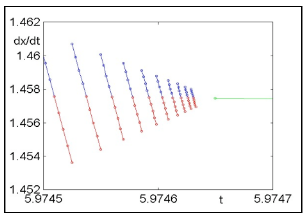
Simulation of systems with chattering

Nonsmooth law at complete chattering

$$x^+ = x^- + \frac{1}{1-r}\left(\frac{2F(x^-)r}{\left(H_x(x^-)F(x^-)\right)_x F(x^-)} + G(x^-)\right)H_x(x^-)F(x^-)$$

$$t^+ = t^- + \frac{1}{1-r}\frac{2H_x(x^-)F(x^-)r}{\left(H_x(x^-)F(x^-)\right)_x F(x^-)}$$





If you want to know more…

[1] B. Brogliato, *Nonsmooth Mechanics,* Springer-Verlag, 2000

[2] SICONOS webpage: http://siconos.inrialpes.fr

[3] M. di Bernardo, C. Budd, A.R. Champneys, P. Kowalczyk, A.B. Nordmark, G. Olivar, P.T. Piiroinen, "Bifurcations in Nonsmooth Dynamical Systems", *SIAM Review* , 2007 (to appear)

[4] M. di Bernardo, C. Budd, A.R. Champneys, P. Kowalczyk, "Piecewise-smooth dynamical systems: Theory and Applications", Springer-Verlag, 2007

… just to start with …