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Stabilisation of Quantised Systems

Luigi Palopoli

Scuola Superiore S. Anna – Pisa, Italy

luigi@gandalf.sssup.it

HYSCOM
IEEE CSS Technical Committee on Hybrid Systems

Information Society Technologies

Siena, July 19-22, 2005 - Rectorate of the University of Siena

Hycon PhD school
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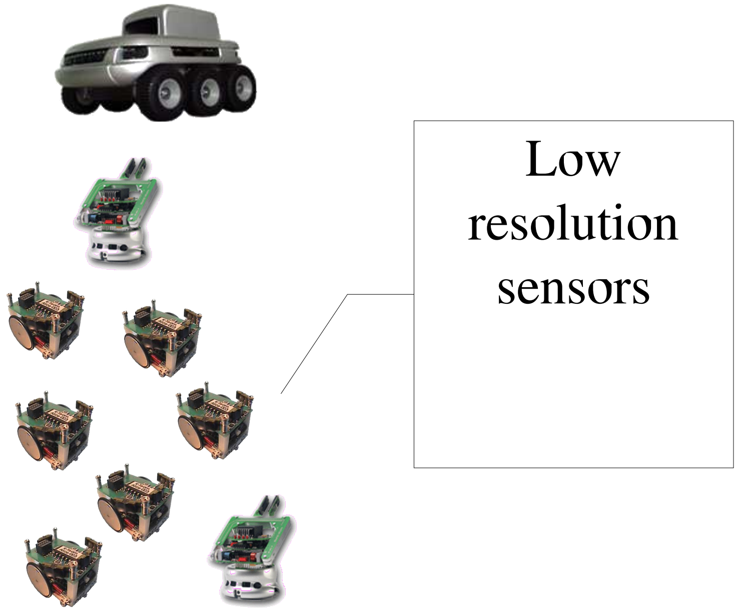
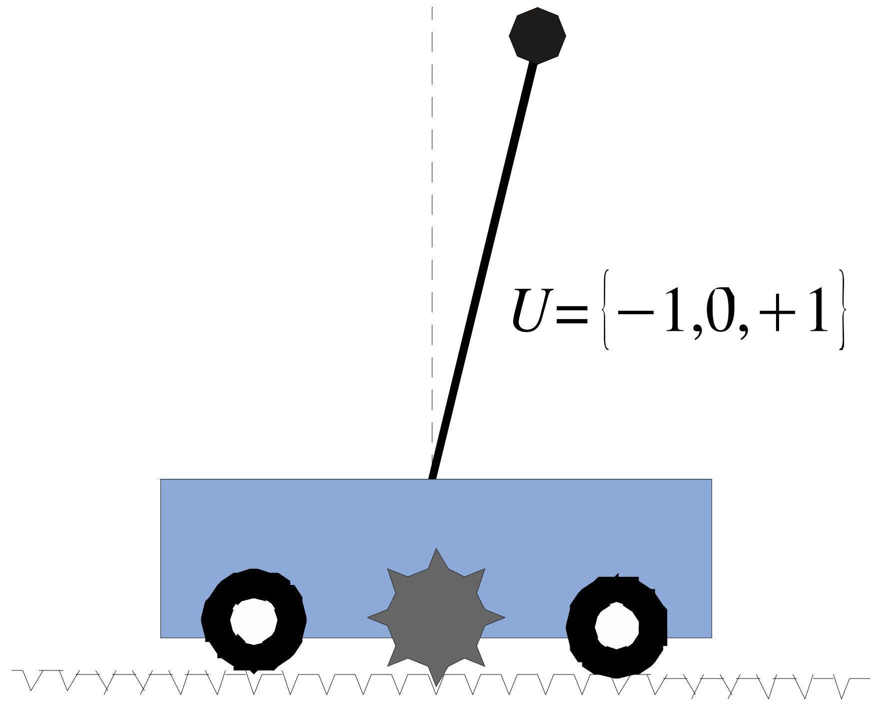
Motivation

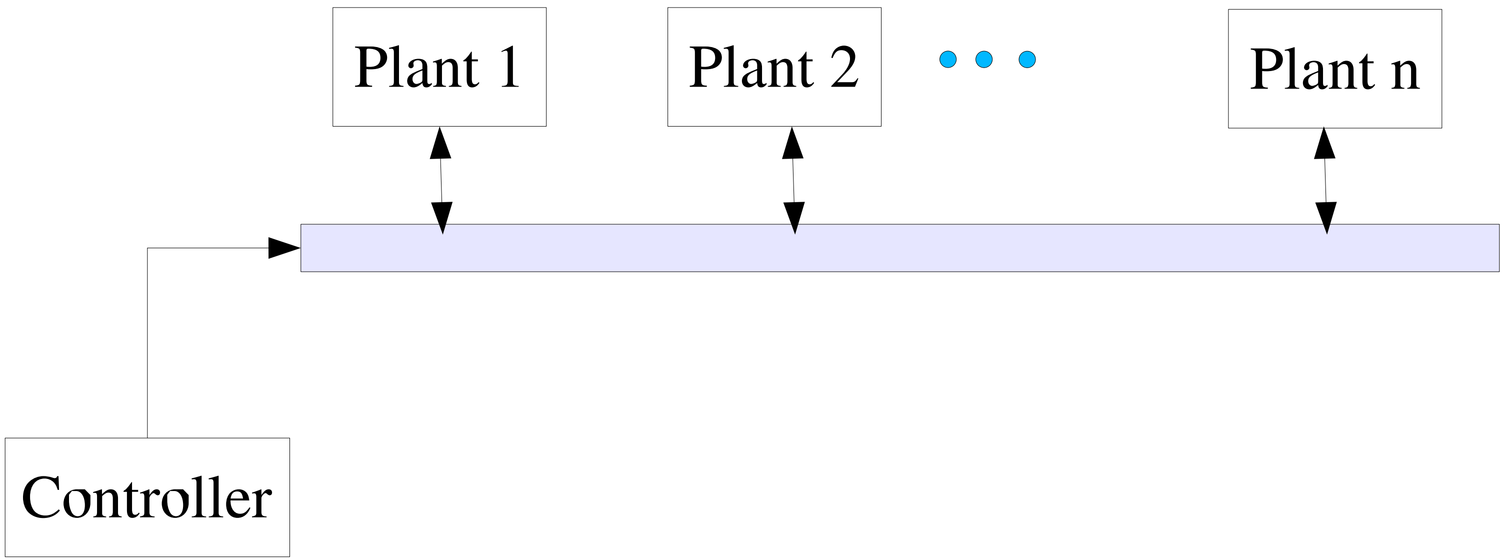
Quantised sensors/actuators





The problem of dealing with quantised resources may arise in practical applications in which a given technology limits the control freedom. The quantiser is imposed.

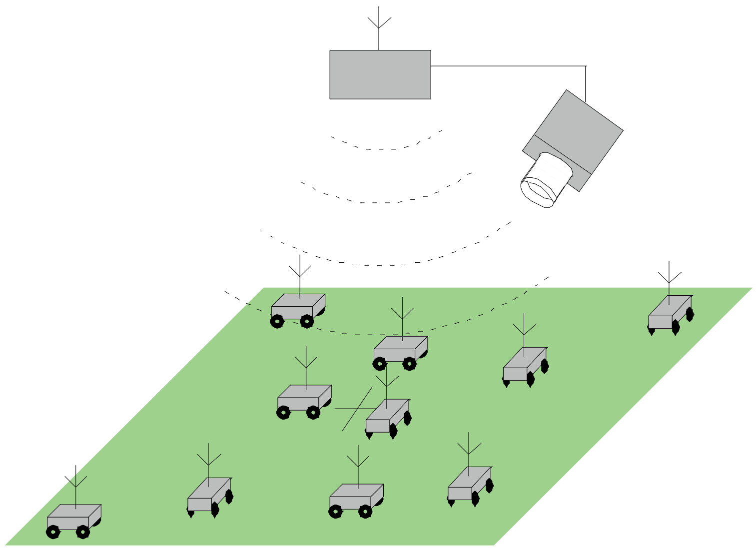
Communication constraints



Control of a large number of systems by a centralised controller:

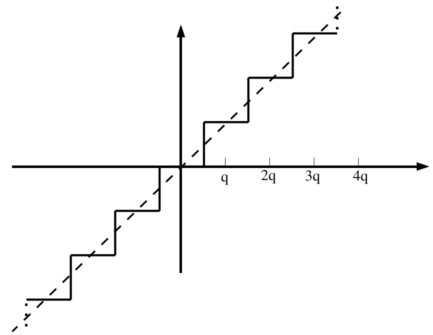
quantisation is instrumental to an efficient communication

Example Scenario



Example: Rendez-vous of multiple vehicles moving on a plane. Each vehicle receives through a communication channel an approximation of its position from a remotely positioned sensor.

Uniform quantisation



- quantisation is often the result of truncation or parameters round-off
  - Digital to Analog conversion at the actuator with a finite resolution (e.g., much coarser than the precision used in the machine).
- typical round-off conversion: $u \longrightarrow q(u) = k$ for $u \in [k - \frac{1}{2}, k + \frac{1}{2}]$ where ϵ is the quantiser's resolution
  - it is also possible to consider a scaled version: $u \longrightarrow \epsilon q_\epsilon(\frac{u}{\epsilon})$
- this quantiser guarantees $|q_\epsilon(u) - u| < 0.5\epsilon$ and it spans a set of uniformly spaced points: $u \in \mathcal{U} = \epsilon\mathbb{Z}$

Logarithmic quantisation

- Recently other schemes have been proposed to the purpose of saving communication bandwidth
- One of the most appealing is logarithmic quantiser:
  - when we are far off from the target we don't need very much precision
- A quantiser of this kind is characterised by: $|q(u) - u| \leq \delta |z|$
- this quantiser spans a set $u \in \mathcal{U} = \{\pm \delta^n u_0, \delta > 1, n \in \mathbb{N}, u_0 > 0\}$

Practical stabilisation of discrete-time linear system with inputs/outputs in discrete sets (fixed quantisation)

Problem formulation

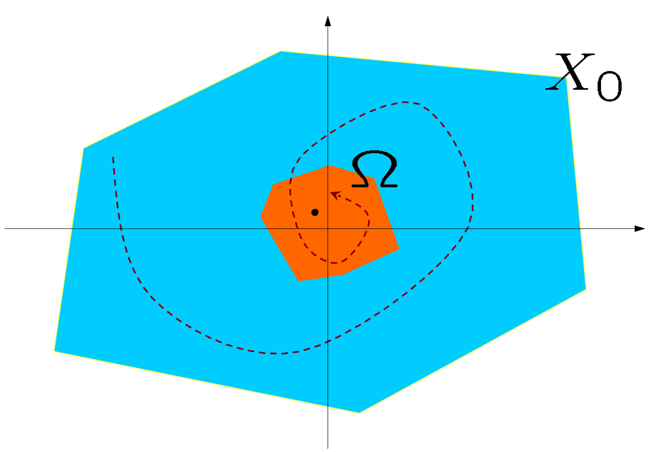
- consider a discrete time system

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= q(x) \end{aligned} \tag{1}$$

  where $u \in \mathcal{U}$ and $y \in \mathcal{Y}$
- assume that the discrete sets U and Y are *given* (for instance they could be imposed by technological limitations of sensors or actuators)
- we want to know:
  1. is it possible to stabilise the system “in some sense”?
  2. what kind of control law do we need to achieve stabilisation?

(X_0, Ω)-Stability

- Back in 1990, Delschamps has proved that exact stabilisation is not attainable
- a better suited notion for quantised control systems (QCS) is practical stability
  - The target "equilibrium point" is a set Ω, which is guaranteed to be controlled invariant
  - The state is assumed to initially lie in an outer set X_0
  - we want the trajectories never to leave X_0 and eventually fall into Ω

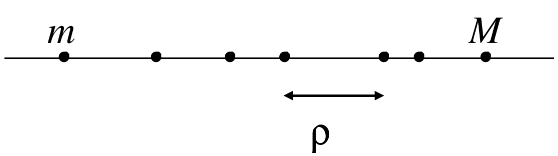


Definitions

Consider a system A, b with inputs in the discrete (and possibly finite) set $\mathcal{U}$

1. the set Ω is controlled invariant if $\forall x \in \Omega$ there exists $u \in \mathcal{U}$ s.t. $x^+ \in \Omega$
2. the system is (X_0, Ω)-stable $\forall x_0 \in \Omega$ there exists N and a sequence of commands $u_0, u_1, \ldots, u_{N-1}$, s.t., 1) $x_k \in X_0$ and $u_k \in \mathcal{U}$ for $k = 1, ..., N$, 2) $x_N \in \Omega$

we aim at finding conditions that allow us to enforce the two conditions above.
The quantiser is identified by a triple (m, M, ρ) :



The controller form

- Picasso and Bicchi (2002) have shown the convenience of:
  - considering systems in standard controller form coordinates
  - considering hypercubic sets $Q_n(\Delta)$ – centred in the origin and of size Δ – for reachability and invariance
- using the control canonical coordinates the evolution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \to \begin{bmatrix} x_2 \\ x_3 \\ \dots \\ \sum \alpha_i x_i + u \end{bmatrix} \tag{2}$$

Key observation: Except for the last component, the evolution of the state is dictated by a shift register. Hence, $x_i \in [-l, +l] \to x_{i-1}^+ \in [-l, +l]$ for $i = 2, \dots, n$

Controlled invariance

- **Theorem (Picasso and Bicchi-2002)** Let A, b be in control canonical form and α_i be the coefficients of the characteristic polynomial and let $a = \sum |\alpha_i|$. Assume that $u \in \mathcal{U}$ characterised by the triple (m, M, ρ) and $\sum |\alpha_i| > 1$

  Then $Q_n(\Delta)$ is controlled invariant iff: $\begin{cases} m \leq -\frac{\Delta}{2}(a-1) \\ M \geq \frac{\Delta}{2}(a-1) \\ \rho \leq \Delta \end{cases}$

Proof

- For the consideration above we have to take care only of the $n - th$ coordinate

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- The controlled invariance of the interval can be imposed as follows:

$$x_n(k+1) \in [-\Delta/2, \Delta/2] \, \forall x(k) \in [-\Delta/2, \Delta/2] \leftrightarrow$$

$$\forall x(k) \in [-\Delta/2, \Delta/2] \exists u \in \mathcal{U} s.t.$$

$$-\Delta/2 \leq x_n(k+1) = \sum \alpha_i x_i(k) + u(k) \leq \Delta/2 \leftrightarrow$$

$$-\Delta/2 - \sum \alpha_i x_i(k) \leq u(k) \leq \Delta/2 - \alpha_i x(k)$$

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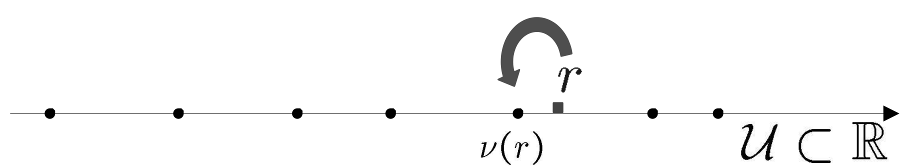
- The segment of acceptable $u(k)$ is Δ, so the quantiser grain has to be $\epsilon \leq \Delta$
- Likewise, the maximum and minimum required values are, in turn, $m \leq -\frac{\Delta}{2}(a-1)$, $M\frac{\Delta}{2}(a-1)$

The feedback law

- Recall that $u(k) = -\sum \alpha_i x_i$ is the *deadbeat* controller
- consider a fixed quantisation scheme with granularity ρ
- The controlled invariance of the interval can be imposed by using:

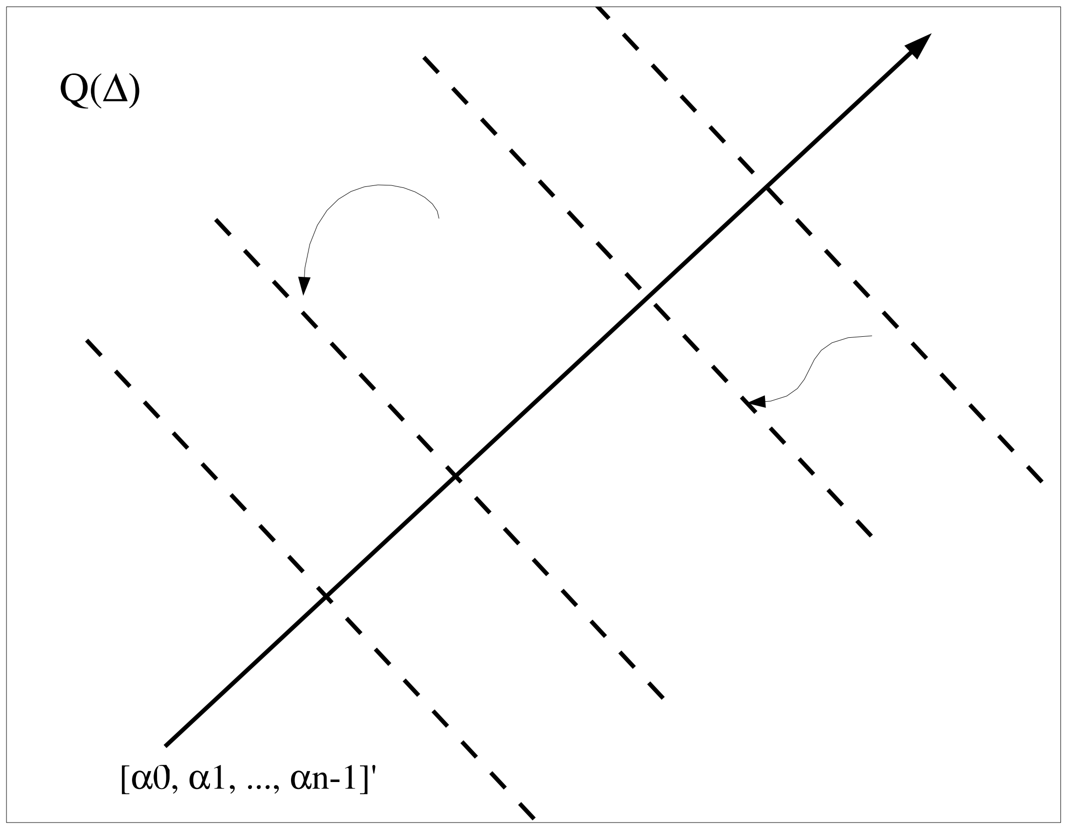
$$-\rho/2 - \sum \alpha_i x_i(k) \leq u(k) \leq \rho/2 - \alpha_i x(k)$$

- we have got only one value ensuring invariance, and this is the quantised version of the deadbeat controller

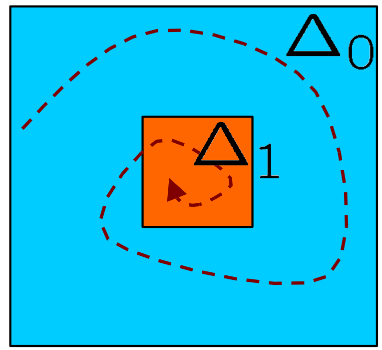


Quantised deadbeat

The quantised dead beat yield a quantisation partition that results into cutting $Q(\Delta)$ by hyperplanes orthogonal to $[\alpha_0, \ldots, \alpha_{n-1}]^T$ (each associated to a quantisation level).



Convergence



- **Theorem (Picasso and Bicchi-2002)** Let A, b be in control canonical form and α_i be the coefficients of the characteristic polynomial and let $a = \sum |\alpha_i|$. Assume that $u \in \mathcal{U}$ characterised by the triple (m, M, ρ) and $\sum |\alpha_i| > 1$, and let $\Delta_0 > \Delta_1 > 0$. Then the system is

$$(Q_n(\Delta_0) - Q_n(\Delta_1)\text{-stabilisable if: } \begin{cases} m \leq -\frac{\Delta_0}{2}(a-1) \\ M \geq \frac{\Delta_0}{2}(a-1) \\ \rho \leq \Delta_1 \end{cases}$$

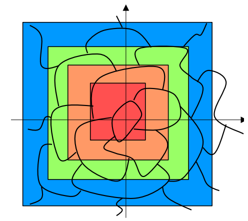
Corollaries

- for a uniform quantiser of resolution ϵ, the system is $(Q_n(\Delta) - Q_n(\epsilon))$-stabilisable in *at most* n steps. The control law attaining stabilisation is the quantised dead-beat: $u(x) = \left\lfloor \frac{\sum \alpha_i x_i + \epsilon/2}{\epsilon} \right\rfloor \epsilon$
- consider for a logarithmic quantiser with symbols:

$$\mathcal{U} = \{0\} \bigcup \{\pm\delta^n u_0,\ s.t. n \in \mathbb{N},\ \delta > 1,\ u_0 > 0\}.$$

If $1 < \delta < \frac{||A||_\infty + 1}{||A||_\infty - 1}$, then $\forall \Delta_0 > u_0$ the q.d.b. controller is $(Q_n(\Delta_0), Q_n(u_0))$-stabilising.

Extension 1: the case of quantised output



- **Theorem (Picasso and Bicchi-2003)** Consider the system:

$$\begin{cases} x^+ = Ax + Bu \\ y(t) = q(x(t)) \\ u \in \mathcal{U} \subset \mathbb{R},\ y \in \mathcal{Y} \subset \mathbb{R}^n \end{cases}$$

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$$\begin{cases} m \leq -\frac{\Delta_0}{2}(a-1) \\ M \geq \frac{\Delta_0}{2}(a-1) \\ \rho + H \leq \Delta_1 \end{cases}$$

where H is a computable parameter of the map q

Extension 2: continuous time and bounded noise

- Consider a continuous time system:

$$\dot{\tilde{x}}(t) = a\tilde{x}(t) + u(t) + w(t), \tilde{x}(0) = x_0, w(t) \in [-w, w]$$

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- The sampled-data equivalent is given by:

$$x(k+1) = \Phi x(k) + \Gamma u(k) + w(k)$$

where $\Phi = e^{aT}$, $\Gamma = \int_0^T e^{as} ds$,
$w(k) = \int_{kT}^{(k+1)T} e^{(a(k+1)T-s} w(s) ds$

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- Controlled invariant interval $I(\Delta)$: for each point there must exist a control value that makes the state confined in $I(\Delta)$ throughout the whole sampling period.

Extension 2: continuous time and bounded noise

- **Preliminary question:** if an interval is controlled invariant for the discrete-time equivalent is it so also for the continuous-time evolution (i.e., what does the state do in the inter-sampling)?

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$$\dot{\tilde{x}}(t) = a\tilde{x}(t) + u(t) + w(t), \tilde{x}(0) = x_0, w(t) \in [-w, w]$$

and assume that it is controlled by a ZoH with sampling period T. An interval $I(\Delta) = [-\Delta/2, \Delta/2]$ is controlled invariant if and only if the discrete time equivalent is.

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- **Proposition (Picasso, Palopoli et al.):** consider the first order system above and assume that $u \in \mathcal{U}$. Then
  1. if $a < 0$, $I(\Delta)$ is controlled invariant iff $\Delta \geq \min\{\frac{\Gamma w}{1-\Phi}; \Gamma(\epsilon + w)\}$

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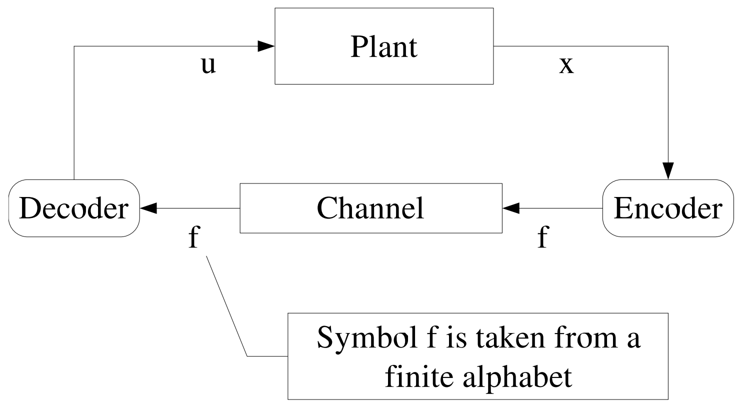
  2. if $a \geq 0$, $I(\Delta)$ is controlled invariant iff $\Delta \geq \Gamma(\epsilon + w)$
- **Remark:** Because the system is affected by noise, the controlled invariance problem is non-trivial also for the case of open loop stable pole.

Control with communication constraints

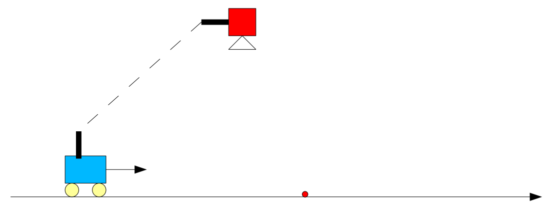
The problem



- So far we have studied the performance of a system when a fixed quantiser is in place
- Another situation of fundamental importance is when quantisation is imposed by communication constraints (e.g., in distributed control systems).
- in general, an encoder/decoder pair and a channel that we will assume noiseless and loss-free, are used in the feedback
- the problem is: what is the minimum bitrate and an encoder/decoder pair to achieve “practical” stabilisation?

An illustrative example (Fagnani and Zampieri 2004)

Assume we want to stabilise a unidimensional vehicle by using a remote sensor. The sensor transmits the position of the vehicle by means of a wireless channel.



Dynamics: $x^+ = x + u$

How do we do it with a few bits?

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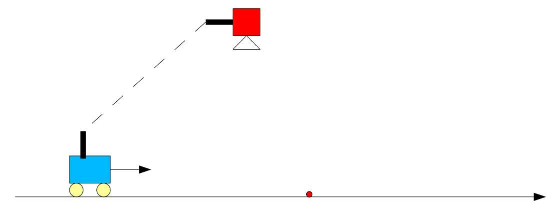
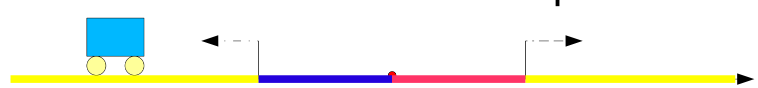


Dynamics: $x^+ = x + u$

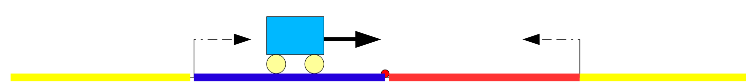
How do we do it with a few bits?

We partition the state space in three areas, and we only say which area we are lying in.

Moreover, we enlarge or shrink the resolution of the quantiser.



Vehicle out of reach: sensor zoom out,
controller don't move



Vehicle in zoom area: sensor zoom in,
controller move toward the target with velocity
proportional to the zooming factor

An illustrative example (Fagnani and Zampieri 2004)

- Sensor (Encoder):

$$(y,\, s_s^+) = \begin{cases} (y_o,\, s_s - 1) & \text{if } |x| > \delta^{s_s} \\ (y_-,\, s_s + 1) & \text{if } -\delta^{s_s} \leq x < 0 \\ (y_+,\, s_s + 1) & \text{if } 0 < x \leq \delta^{s_s} \end{cases}$$

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- Controller at the vehicle (Decoder):

$$(u,\, s_v^+) = \begin{cases} (0,\, s_v - 1) & \text{if } y = y_o \\ (0.5\delta^{s_v},\, s_v + 1) & \text{if } y = y_- \\ (-0.5\delta^{s_v},\, s_v + 1) & \text{if } y = y_+ \end{cases}$$

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- At the initial instant we have to synchronise the zooming factors at the controller and at the sensor: $s_s = s_v$.

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- At the initial instant we have to synchronise the zooming factors at the controller and at the sensor: $s_s = s_v$.
  - this relation will always be maintained because the channel is ideal

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- At the initial instant we have to synchronise the zooming factors at the controller and at the sensor: $s_s = s_v$.
  - this relation will always be maintained because the channel is ideal
- The system is asymptotically stable if $\delta \geq 0.5$. Indeed, in this case, we can write: $|x| \leq \delta^s \rightarrow |x^+| \leq 0.5\delta^s \leq \delta^{s+1} = \delta^{s^+}$. Therefore at very state we will shrink the state.

An illustrative example (Fagnani and Zampieri 2004)

- Sensor (Encoder):

$$(y,\, s_s^+) = \begin{cases} (y_o,\, s_s - 1) & \text{if } |x| > \delta^{s_s} \\ (y_-,\, s_s + 1) & \text{if } -\delta^{s_s} \leq x < 0 \\ (y_+,\, s_s + 1) & \text{if } 0 < x \leq \delta^{s_s} \end{cases}$$

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- One potential problem is that the encoder/decoder pair have to maintain a perfectly synchronised state information (even one simple packet loss could cause instability)
- Moreover the number of states of the encoder/decoder state is infinite (albeit denumerable)

Performance/complexity tradeoffs

- Fagnani and Zampieri (2003, 2004) propose to consider, in the general case, the following problem: how can we relate the closed loop performance to the controller complexity.
- System:

$$\begin{cases} x^+ = Ax + Bu \\ y = Gx \end{cases}$$

- Controller:

$$\begin{cases} s^+ = f(s, y) \\ u = k(s, y) \end{cases}$$

where, $s \in S$ with S finite or denumerable, and the maps $k(s, .)$ and $f(s, .)$ are quantised for each s, i.e., there exist two finite partitions $\mathcal{K}_s = \{K_s^1, \ldots, K_s^{N_s}\}$ and $\mathcal{F}_s = \{F_s^1, \ldots, F_s^{N_s}\}$ of the $\mathbb{R}^p$ (p is the dimension of y) such that

  - $\bigcup K_s^j = \mathbb{R}^p$,$\bigcup F_s^j = \mathbb{R}^p$
  - $k(s, .)$m $f(s, .)$ are constant respectively in each partition K_s^j and F_s^j

Performance/complexity tradeoffs

- **Performance parameters:** Considering the problem of (W, V)-stability the we consider:
  - the contraction rate $C = \lambda(W)/\lambda(V)$, where $\lambda()$ is the Lebesgue measure
  - the mean time T used for reducing the state from W to V
- **Performance parameters:**
  - L number of states of the controller (utilised for the reduction form W to V)
  - N maximum number of the partitions $\mathcal{K}_s$ over s
  - M maximum number of the partitions $\mathcal{F}_s$ over s

Memoryless uniform quantisation

Recall that

- the system is $(Q_n(\Delta_0) - Q_n(\Delta_1)$-stabilisable if: $\begin{cases} m \leq -\frac{\Delta_0}{2}(a-1) \\ M \geq \frac{\Delta_0}{2}(a-1) \\ \rho \leq \Delta_1 \end{cases}$ where $a = \sum \alpha_i$. That means that the minimum number of levels is:

$$N = \left\lceil a \frac{\Delta_0}{\Delta_1} \right\rceil = \left\lceil a C^{\frac{1}{n}} \right\rceil$$

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Therefore, for large C, we get: $N \approx aC^{\frac{1}{n}}, \ T \approx n$ which shows that for large C the entrance time does not depend on C.

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- The computation of the mean entrance time is a little bit more involved

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$$N \approx \frac{a}{\delta n}\frac{\log C}{\log \delta^{-1}}, T_r \approx \frac{\overline{T}}{n}\frac{\log C}{\log \delta^{-1}}$$

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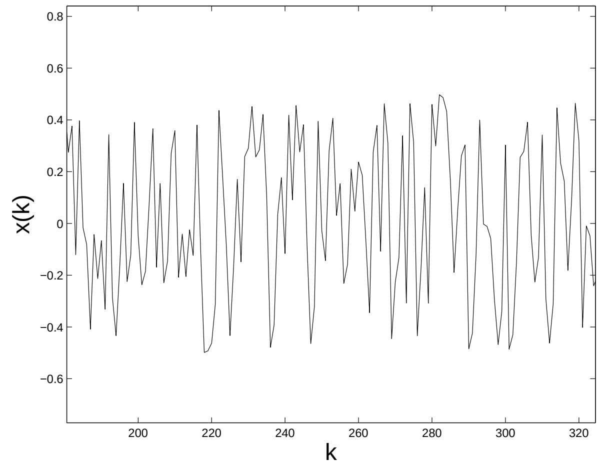
- Consider a scalar system $x^+ = ax + u$ and assume that we have a controller that makes an interval Δ controlled invariant
- How do the trajectory move inside the invariant?
- If we choose the quantised control law appropriately we can inject an ergodic behaviour for almost all initial points (Zampieri and Fagnani 2003). Thereby, by simply making the V invariant we can have (V-W)-stability.

Chaotic controller



- using a chaotic scheme we can have a number of levels

$$N = 2\left\lceil |a| \right\rceil$$

  independent of the contraction rate!

- Clearly we must have time to wait:

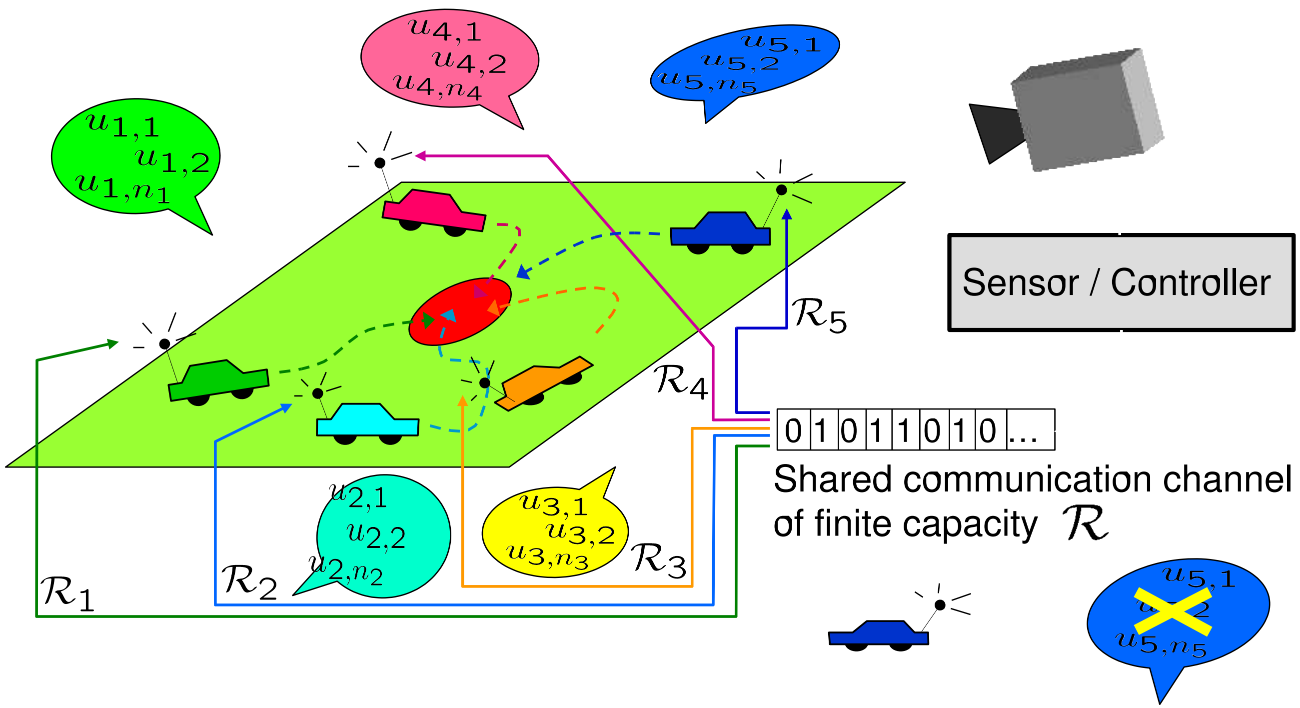
$$T \approx C \log C$$

An application Design Example
Picasso-Palopoli et al. 2004

A motivating example

In a distributed control problem we can encounter both sources of quantisation:

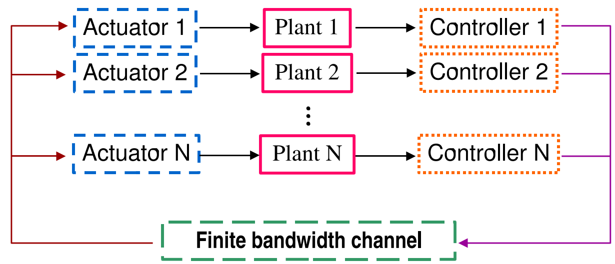
- low cost sensors/actuators
- finite communication bandwidth on shared channels



Problems:

1. which quantisation level on each vehicle should we utilise?
2. how should we distribute the shared channel capacity

Problem formulation



- Consider a set of linear and first order systems $\dot{x}_i = a_i x_i + u_i + w_i$

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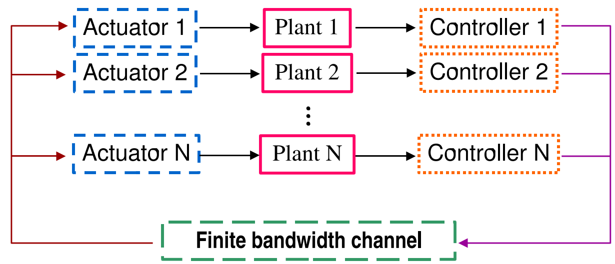

- Consider a set of linear and first order systems $\dot{x}_i = a_i x_i + u_i + w_i$
- Assumptions:
  1. controls are quantised: $u_i \in \epsilon\mathbb{Z}$
  2. the channel has a finite capacity $\mathcal{R}$, which is *statically* allocated amongst the different systems.
  3. the noise is bounded: $w_i(t) \in [-w/2, w/2]$
  4. we use a fixed sampling period and piecewise constant control (Zoh); moreover, the feedback law is *memoryless*

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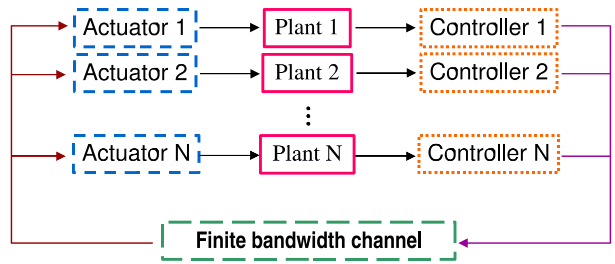

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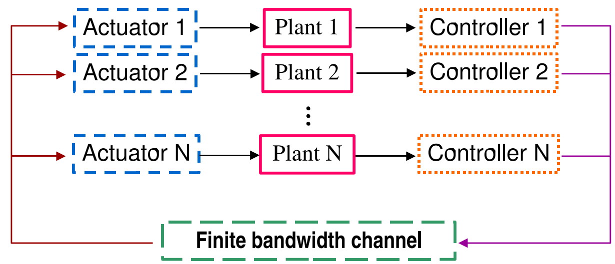

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- Control goal: achieve practical ($(I(\Delta_i), I(\delta_i))$-stability) on each control loop, where $I(x) = [-x/2, x/2]$
- Design parameters: R_i bitrate assigned to the i-th system, Sampling periods T_i, control sets $\mathcal{U}_i \subseteq \epsilon_i\mathbb{Z}$

The envisioned methodology - I

- Let $\mathbf{\Delta}$ and δ be vectors of reals such that the i-th system is $(I(\Delta_i), I(\delta_i))$-stable; let $\mathbf{\Delta}_0$ and δ_0 respectively denote the minimum and the maximum required values

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arg\min_{\mathbf{R},\mathbf{T},\mathbf{\Delta},\delta} f(\mathbf{\Delta},\delta)
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\text{subj. to} \begin{cases} \mathbf{\Delta} \geq \mathbf{\Delta}_0 \\ \delta \leq \delta_0 \\ \sum R_i \leq \mathcal{R} \\ (\mathbf{R}, \mathbf{T}, \mathbf{\Delta}, \delta) feasible \end{cases}
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- the analysis allows us to identify the minimum bitrate $R^{(i)}_{min}(\Delta_i, \delta_i)$ to attain the specification (Δ_i, δ_i). The problem is simplified as

$$arg\min_{\mathbf{\Delta},\delta} f(\mathbf{\Delta}, \delta)$$

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The envisioned methodology - I

- we solve

$$arg\min_{\mathbf{\Delta},\delta} f(\mathbf{\Delta},\delta)$$
$$\text{subj. to} \begin{cases} \mathbf{\Delta} \geq \mathbf{\Delta}_0 \\ \delta \leq \delta_0 \\ \sum R_{min}^{(i)}(\Delta_i,\delta_i) \leq \mathcal{R} \end{cases}$$

  by numeric optimisation techniques coming up with an optimal solution $(\mathbf{\Delta}^*,\delta^*)$

- from the optimal bitrate R^* we can reconstruct the optimal sampling period T_i^* and the optimal set of controls $\mathcal{U}_i^*$

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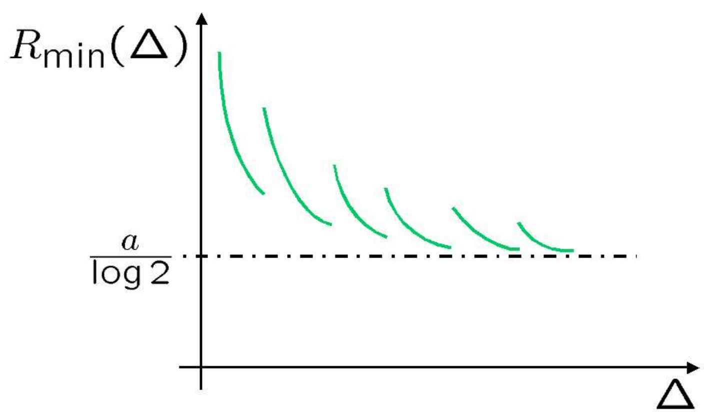
- $R_{min}(\Delta, \delta)$ can simply be found as

$$R_{min}(\Delta, \delta) = \min_T \rho(\Delta, \delta, T)$$

Explicit results

- The general problem can be numerically solved for certain classes of quantisation policies (e.g., simulation of a logarithmic quantiser on a fixed one)
- If we consider only controlled invariance of the target set δ, there are stronger (explicit) results
- For unstable plants:

$$R_{min}(\Delta) = \frac{a}{\log\left(1 + \frac{a\Delta}{2\epsilon\left\lceil \frac{a\Delta + w}{2\epsilon}\right\rceil + w}\right)}$$



An example problem

As an example problem we considered the following

$$arg\min_{\delta} |\delta_0 - \delta|_\infty$$
$$\text{subj. to} \sum R^{(i)}_{min}(\delta_i) \leq \mathcal{R}$$

Because R_{min} is not analytic, the feasibility region is disconnected and not convex.

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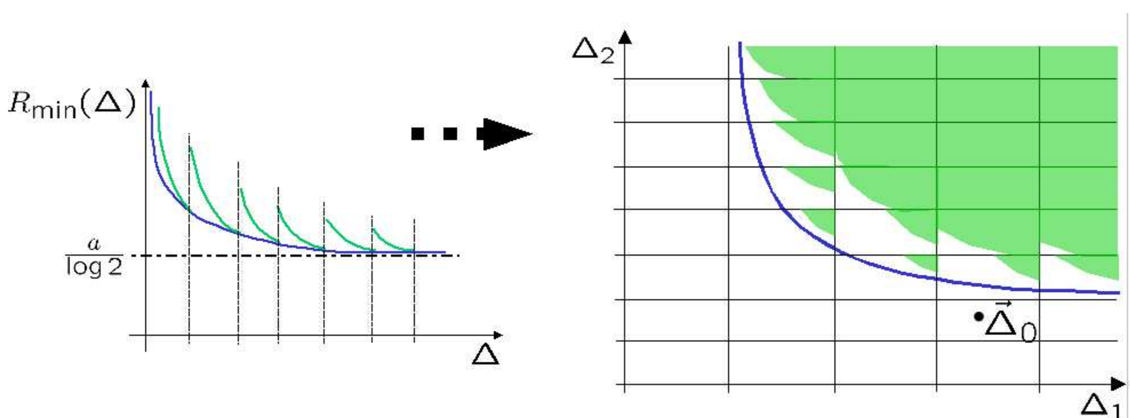
Because R_{min} is not analytic, the feasibility region is disconnected and not convex.

However, we can use an analytic lower bound of R_{min} and come up with an easy-to-compute lower bound of the problem.

$$\underline{R}_{min}(\delta) = \frac{a}{\log\left(1 + \frac{a\delta}{a\delta + 2(w+\epsilon)}\right)}$$



An example problem - I

- Generally speaking, the lower bound can be found by finding the zero of a non-linear equation.
- If $\frac{w_i}{a_i \delta_i} \gg 1$ the expression of the lower bound is particularly neat:

$$\delta_h^* \approx \frac{2 \sum_i (w_i + \epsilon_i)}{\mathcal{R} - \sum a_i}$$

$$R_h^* \approx a_h + \frac{w_h + \epsilon_h}{\sum (w_i + \epsilon_i)} (\mathcal{R} - \sum a_i)$$

- The exact solution can be found by using a Branch and Bound scheme

Conclusions

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- Control with quantisation has become a very active research field in the last few years.
- In this talk we have briefly surveyed some results related to the problem of stabilisation
- Other approaches consider quantisation from a more closely information theoretical point of view (Delschamps, Nair-Evans), or using model predictive control (Picasso-Bemporad-Bicchi)
- Another very active research area is to use quantisation in planning, verification and design problems
  - lattice based analysis/synthesis for discrete-time nonholonomic systems (Bicchi-Marigo-Piccoli)
  - discrete bisimulations (Tabuada-Pappas)
- ...and we just scraped the surface!

Some reference

References

[1] N. Elia and S. Mitter, *Stabilization of Linear Systems With Limited Information* , IEEE Trans. Autom. Control, 46(9); pages: 1384–1400. 2001.

[2] F. Fagnani and S. Zampieri *Stability analysis and synthesis for scalar linear systems with a quantized feedback* , *IEEE Trans. Automat. Control*, AC-48:1569–1584, 2003.

[3] F. Fagnani and S. Zampieri, *Quantized stabilization of linear systems: complexity versus performance* , `To appear` on Transactions on Automatic Control, special issue on “Networked Control Systems”. 2004.

[4] B. Picasso, F. Gouaisbaut and A. Bicchi, *Construction of invariant and attractive sets for quantized–input linear systems* , Proc. of the 41st IEEE Conference on Decision and Control “CDC 2002” pages: 824–829. 2002.

[5] S.C. Tatikonda, *Control under communication constraints* , Ph.D. thesis, Massachusetts Institute of Technology. 2000.

[6] B. Picasso, L.Palopoli, A. Bicchi and K.H Johansson *Control of distributed embedded systems in the presence of Unknown-but-bounded Nois* , Control and decision conference (cdc04). 2004.