1st HYCON PhD School on Hybrid Systems

www.unisi.it

www.ist-hycon.org

Modeling and Control of Stochastic Hybrid Systems

John Lygeros
University of Patras, Greece

lygeros@ee.upatras.gr

Hybrid systems combine continuous dynamics (differential or difference equations) typical of physical plants and discrete dynamics (automata and logical conditions) typical of control logic. By combining disciplines of computer science and systems and control theory, research on hybrid systems provide a solid theory and computational tools for the analysis, simulation, verification, and control design of "embedded systems", and are used in a large variety of applications (automotive systems, air traffic management, biological systems, process industries, and many others).

HYSCOM
IEEE CSS Technical Committee on Hybrid Systems

Information Society Technologies

Siena, July 19-22, 2005 - Rectorate of the University of Siena

Modeling and control of stochastic hybrid systems

John Lygeros

Department of Electrical and Computer Engineering
University of Patras
Rio, Patras, GR-26500
Greece
lygeros@ee.upatras.gr

Siena, July 21, 2005

Hybrid systems

Dynamical systems with discrete and continuous state and/or input variables

$$q \in Q = \{q_1, q_2, q_3\}$$

$$x \in \mathbb{R}^n$$

q changes discretely

$$q(t^-) \mapsto q(t^+)$$

x changes either discretely, or continuously

$$x(t^-) \mapsto x(t^+)$$

$$\dot{x}(t) = f(x(t), q(t))$$

Hybrid Automata [Johansson et.al.]

Hybrid automaton:

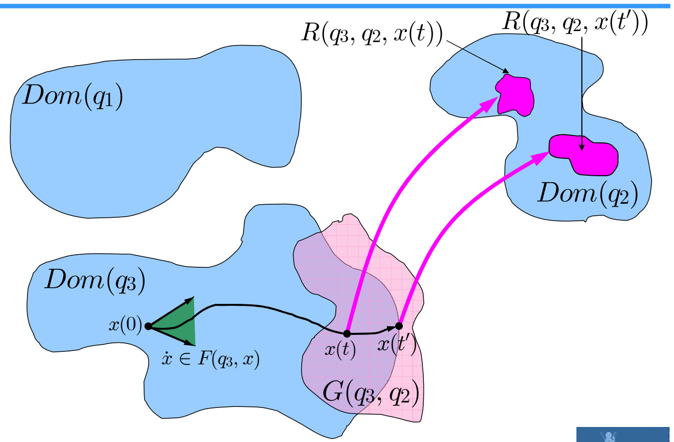
$$H = (Q, X, Init, F, Dom, E, G, R)$$

- Discrete state variables $Q = \{q_1, q_2, q_3, \ldots\}$
- Continuous state variables $X = \mathbb{R}^n$
- Initial conditions $Init \subseteq Q \times X$
- Continuous dynamics $F : Q \times X \to 2^{\mathbb{R}^n}$
- Domain of continuous evolution $Dom : Q \to 2^X$
- Discrete transitions $E \subseteq Q \times Q$
- Guards $G : E \to 2^X$
- Transition relation $R : E \times X \to 2^X$

Rough interpretation

- 2^X power set (set of all subsets) of *X*
- State of the system $(q, x) \in Q \times X$
- Start with $(q, x) \in Init$
- Continuous motion $\dot{x} \in F(q, x)$ …
- … provided that $x \in Dom(q)$
- Discrete transition $q \mapsto q'$ only if
  - $(q, q') \in E$
  - $x \in G(q, q')$
- After discrete transition $x' \in R(q, q', x)$

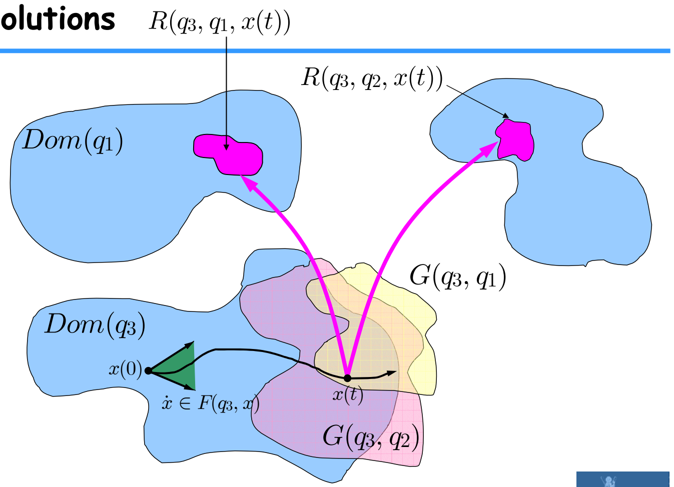
Solutions



Uncertainty

- Solutions defined "declaratively" (cf. "imperatively")
- Allows uncertainty
  - Select any $(q, x) \in Init$
  - Multiple continuous flow directions $\dot{x} \in F(q, x)$
  - Multiple discrete state destinations
    $$G(q, q') \cap G(q, q'') \neq \emptyset$$
  - Multiple continuous state destinations
    $$x' \in R(q, q', x)$$
  - Choice between flowing and jumping
    $$x \in Dom(q) \cap G(q, q')$$

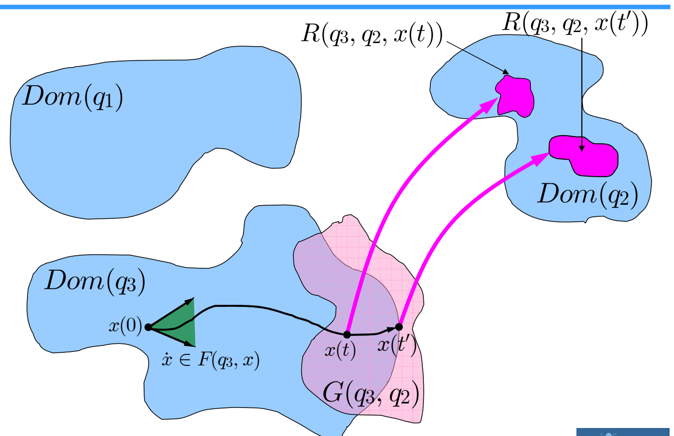
Solutions



Uncertainty

- Solutions defined "declaratively" (cf. "imperatively")
- Allows uncertainty
  - Select any $(q, x) \in Init$
  - Multiple continuous flow directions $\dot{x} \in F(q, x)$
  - Multiple discrete state destinations
    $$G(q, q') \cap G(q, q'') \neq \emptyset$$
  - Multiple continuous state destinations
    $$x' \in R(q, q', x)$$
  - Choice between flowing and jumping
    $$x \in Dom(q) \cap G(q, q')$$

Solutions



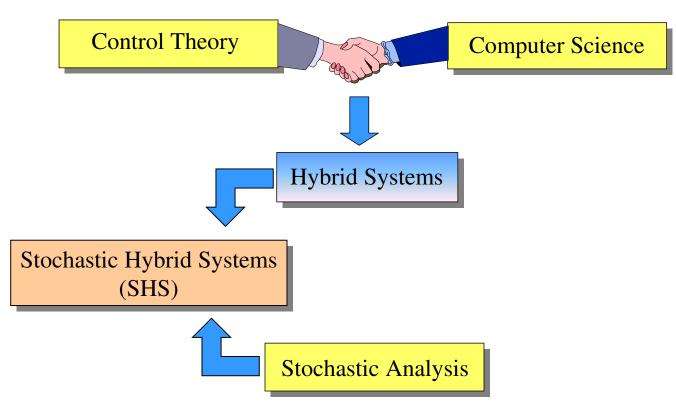
Non determinism

- Choice typically non-deterministic
  - Many solutions possible
  - Nothing to distinguish one from the others
  - E.g. which is likely and which is not
- Only yes/no questions can be answered
- OK model of uncertainty for certain applications
- E.g. robust control
- In some applications finer levels of information needed

Examples and applications

- Communication networks
- Mathematical finance, insurance
- Fault tolerant manufacturing
- Air traffic management
  - Continuous dynamics
  - Discrete dynamics
  - Stochastic dynamics: Wind and weather, malfunctions, actions/errors of humans
  - Accidents will always be possible
  - We would like to reduce their probability
- Biochemical networks

Stochastic Hybrid Framework



Disclaimer

- Very technical field
- In some areas, general understanding just emerging
- Omit technical details, stick to big picture
- Indicate where knowledge ends

- Discussion in continuous time
- In discrete time issues are similar, technical problems are usually simpler

Outline

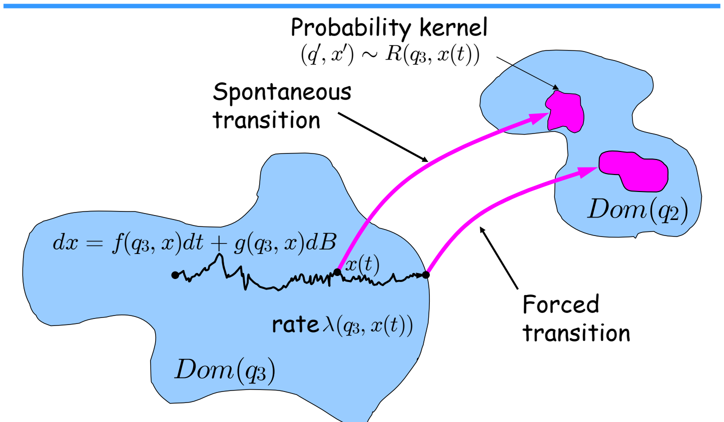
1. Modeling
   1. Classes of models
   2. Comparison
2. Analysis and control
   1. Overview of analysis and control problems
   2. Reachability
   3. Monte-Carlo approximation
3. Applications
   1. Biochemical networks: DNA replication
   2. Air traffic control

1. Modeling

Classes of SHS

- Autonomous or with inputs
  - Modeling and analysis, stability, reachability
  - Composition, optimal control, stabilization, etc.
- Stochastic uncertainty
  - Initial condition: probability distribution
  - Continuous evolution (e.g. SDE)
  - Destination of discrete transitions: probability distribution, depends on state before transition
  - Choice between jumping and flowing
    - Markovian jumps with rates that depend on continuous state (spontaneous transitions)
    - Boundary hits (forced transitions)

Solutions



Classes of SHS

- Ingredients present in different combos
- More general models preferable
  - Properties inherited by special cases
- Fewer properties known for general models
  - Existence, non-Zeno, CADLAG
  - Markov and strong Markov properties
  - Generator of the process
  - Characterization of value functions
  - Invariant distributions, stability
- Important in theory and in practice
  - E.g. theoretical foundations of Monte-Carlo simulation

Modeling frameworks

- To illustrate the issues consider three classes of stochastic hybrid modeling frameworks
  - Piecewise Deterministic Processes (PDP) (Davis 1980's)
  - Switching diffusion processes (SDP) (Ghosh et.al. early 1990's)
  - "Stochastic hybrid systems" (SHS) (Hu et.al. late 1990's)
- All autonomous

PDP: Formal model

Under

- **Assumption**: *X(i) are open, f(i,.) is globally Lipschitz, λ(.) is measurable, λ(.) is locally integrable, R(A,.) is measurable, the process* $N_t = \sum_i I_{(t \geq T_i)}$ *is such that* $E(N_t) < \infty$ *for all times t;*

The PDMP is a **strong Markov** process (Davis 1985), and its expression of infinitesimal generator is given by:

$$L^{PDMP}\theta(\beta) = (\nabla\theta(\beta))^T \cdot f + \lambda(\beta) \int_{D^*(Q,d,X)} R(d\alpha,\beta)(\theta(\alpha)-\theta(\beta))$$

Where θ belongs to the domain of generator as in Davis 1985.

SDP: Formal model

Under

- **Assumption**: *f(i,x),* $\sigma_{ij}(i,x)$*, and* $\lambda_{ij}(x)$ *are bounded and Lipschitz;*

The SDP is a strong **Markov** process (Ghosh et al. 1991), and its expression of infinitesimal generator is given by:

$$L^{SDP}\theta(i,x) = (\nabla\theta(i,x))^T \cdot f + \frac{1}{2}Tr\left(\sigma(i,x)\sigma(i,x)^T\right)H^{\theta}(i,x) + + \sum_{j=1}^{N}\lambda_{ij}(x)(\theta(,x)-\theta(i,x))$$

Where θ belongs to the domain of generator as in Ghosh et al. 1991.

SHS: Formal model

Under

- **Assumption**: *f(i,x) and g(i,x) are bounded and Lipschitz in x, R(.,.) is measurable;*

The SHS is a **Markov** process (Hu et al. 2000), and

Theorem: The SHS expression of infinitesimal generator is given by:

$$L^{SHS}\theta(\alpha) = (\nabla\theta(\alpha))^T \cdot f + \frac{1}{2}Tr\left(\sigma(\alpha)\sigma(\alpha)^T\right)H^{\theta}(\alpha) + + \int_{D^*}(\theta(\beta)-\theta(\alpha))R(d\beta,\alpha)$$

Where θ belongs to the domain of generator.

Classification of some SHS [12]

	PDP [3,4]	SDP [5,6]	SHS [8]	GSHP [2,14]
Stochastic continuous evolution				
Forced transitions & continuous reset				
Spontaneous transitions				

Known properties

- PDP:
  - Well posedness, strong Markov property
  - Generator, domain characterization
  - Optimal control, stability
- SDP
  - Well posedness, strong Markov property
  - Generator, domain characterization
  - Optimal control, stability
- SHS
  - Markov property
- GSHP
  - Well posedness, strong Markov property
  - Generator

2. Analysis and Control

Stability and stabilization

- Different notions of stochastic stability
  - Existence of invariant measures
  - Moment asymptotic stability
  - Almost sure asymptotic stability
  - …
- Sufficient conditions based on Lyapunov functions
- Well studied for classes of SDP
- Studied in the 1980's for PDP
- Very little known about other SHS classes

Optimal control

- Introduce control variables to
  - Drive continuous motion
  - Influence discrete transition rate
  - Force discrete transitions
  - Influence discrete transition destination
- Different combos in different approaches
- Introduce admissible control policies
  - Feedback
  - Markov
  - Non-anticipative
- Introduce cost function to assign cost to control policy

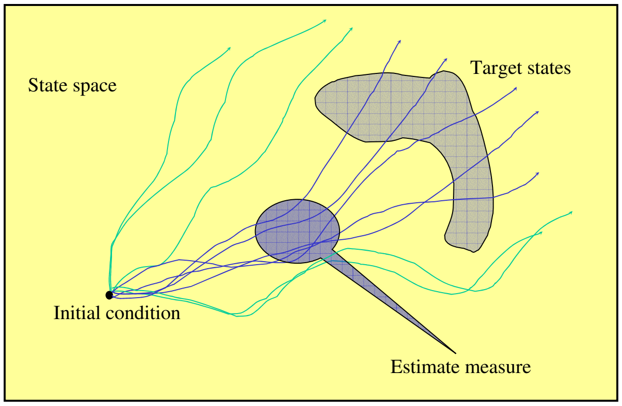
Optimal control

- Usually expected value

$$\mathbb{E}\{\int_0^\infty l(x(t),u(t))dt + \sum_{i=0}^\infty c(x(\tau_i),u(\tau_i))\}$$

- Minimize over all admissible control policies
- Define value function
- Develop dynamic programming principle
- Characterize value function as PDE solution
  - Coupled second order for SDP
  - First order with boundary conditions for PDP
  - ??? for others

Reachability



Reachability

- Underlying probability space (Ω, M, P)
- State space $(\mathbb{X}, B)$
- Stochastic process $x:\Omega\times\mathbb{R}_+\to\mathbb{X}$
- Given $E\in B$ and $T\geq 0$
- Reach "events"

$$Reach_T(E)=\{\omega\in\Omega \mid \exists t\in[0,T] \text{ such that } x(\omega,t)\in E\}$$

$$Reach_\infty(E)=\{\omega\in\Omega \mid \exists t\geq 0 \text{ such that } x(\omega,t)\in E\}$$

- Reach probability

$$P\{Reach_T(E)\}$$

Alternative characterization

- Define indicator function

$$I_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

- Note that

$$I_E(x(t)) = 1 \Leftrightarrow x(t) \in E$$

$$\max_{t\in[0,T]} I_E(x(t)) = 1 \Leftrightarrow \exists t \in [0,T] : \ x(t) \in E$$

$$P\{Reach_T(E)\} = P\{\max_{t\in[0,T]} I_E(x(t)) = 1\}$$
$$= \mathbb{E}\{\max_{t\in[0,t]} I_E(x(t))\}$$

Immediate technical problem

- Is the set Reach_T(E) an event?

$$Reach_T \in M$$

- Equivalently, is

$$\max_{t\in[0,T]} I_E(x(t))$$

a random variable?
- Answer trivially yes in discrete time
- Answer is often yes in continuous time
  - E.g. SDP, PDP
- Technical conditions need to be introduced

Siena, July 21, 2005

Computation of Reach_T

- Analytical estimates [Bujorianu et.al.]
  - Dirichlet forms, potential theory
  - Links to Lyapunov stability
- Computational estimates [Katoen et.al.]
  - Symbolic model checking
  - Restricted classes
- Numerical estimates [Prandini, Tomlin et.al]
  - Numerical solution of PDE's
  - Approximate by Markov chains (next talk)
- Statistical estimates
  - Monte-Carlo simulation

Siena, July 21, 2005

Monte-Carlo simulation

- Assume simulator of SHS is available
- Simulate N times $x^i(\cdot) : [0,T] \to \mathbb{X}$
- Count number of simulations that reach E

$$I_{Reach_T(E)}(x(\cdot)) = \begin{cases} 1 & \text{if } \exists t \in [0,T] : \ x(t) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Estimate

$$\hat{P}^N = \frac{1}{N}\sum_{i=1}^{N} I_{Reach_T(E)}(x^i(\cdot))$$

Siena, July 21, 2005

Convergence of Monte-Carlo

- Under very weak assumptions

$$\lim_{N\to\infty} \hat{P}^N = P\{Reach_T(E)\}$$

- Moreover, given $\delta, \epsilon \in (0,1)$

$$P\{|\hat{P}^N - P\{Reach_T(E)\}| > \epsilon\} \leq \delta$$

provided that $N \geq \frac{1}{2\epsilon^2}\ln(\frac{2}{\delta})$
- "Fast" growth w.r.t. ε slow w.r.t. δ
- Growth still polynomial
- Sample size independent of state dim.
- Simulation time affected by it

Siena, July 21, 2005

Particle implementation

- Each simulation treated as "particle"
- Particles simulated in parallel
- Particles interact at each simulation step
- "Good" particles get rewarded
  - Assigned bigger weight
  - Allowed to produce more offspring
- Advantages
  - Substantial speed-up in rare event simulation
  - Recursive implementation

Siena, July 21, 2005

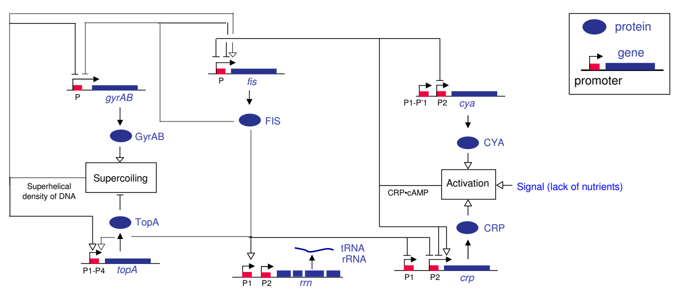
3. Applications

3.1 Biochemical networks

Biochemical network modeling

- Biochemical networks:
  - Describe interactions of genes, proteins and other molecules in the cell
  - Provide information about cell function

Example: stress responce in *E. coli*



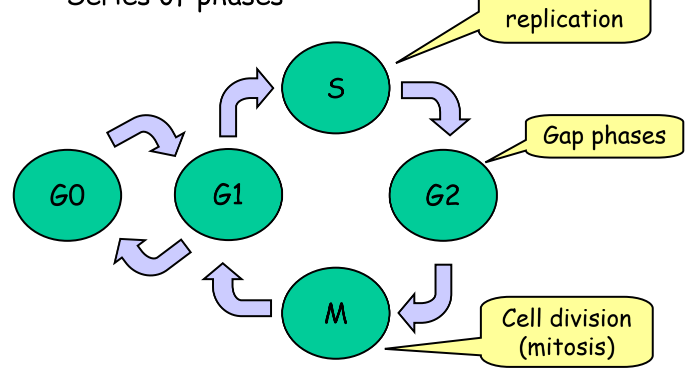
Courtesy: H. de Jong, INRIA

Hybrid dynamics

- Hybrid dynamics at many levels
  - Genes on/off
  - Thresholds of protein concentration
  - Cell cycle phases
  - Cell differentiation
- Develop a simple SHS model of DNA replication in the cell cycle

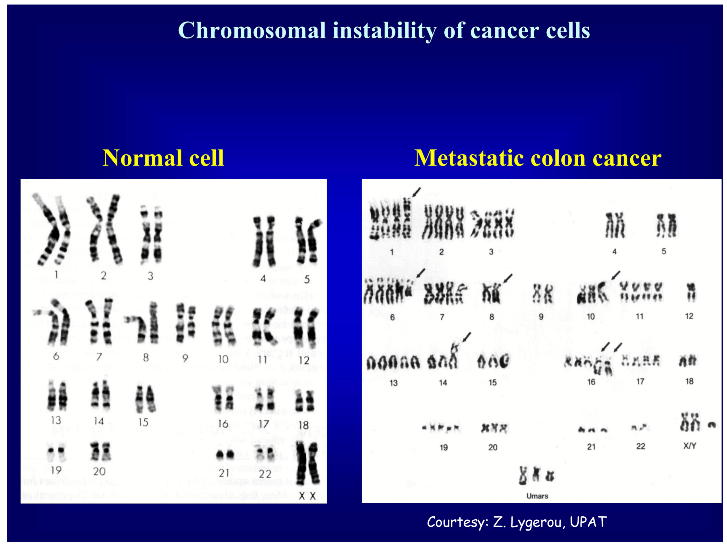
Cell cycle

- Series of phases



Biochemical network

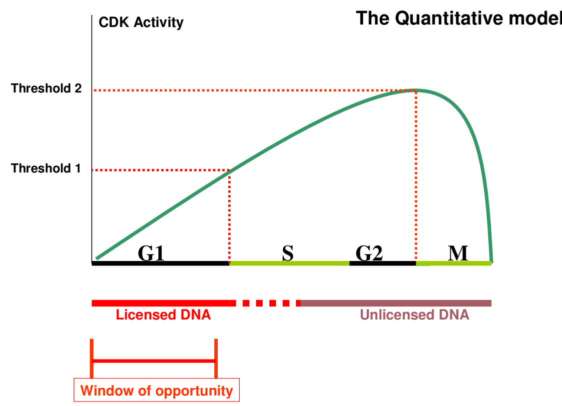
- Progress through the cell cycle needs to be tightly regulated
  - Mitosis without complete S phase => death
  - Multiple S phases without mitosis not good either (linked to cancer)



Biochemical network

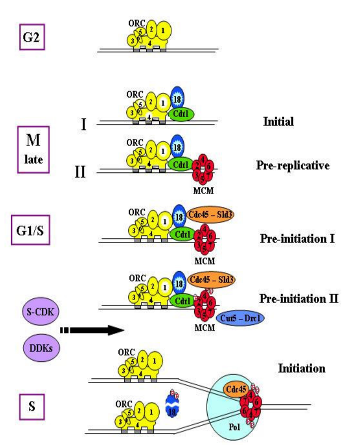
- Progress through the cell cycle needs to be tightly regulated
  - Mitosis without complete S phase => death
  - Multiple S phases without mitosis not good either (linked to cancer)
- Complex biochemical network (~15 molecules involved, Novak et.al.)
- Key ingredient: CDK proteins
- CDK activity fluctuation sets the pace for cell cycle (Nurse et.al.)
- Process of threshold crossing

CDK activity [Nurse et.al.]



S-Phase

- Split DNA and make a copy
- Splitting in many locations along genome
  - Origins of replication
- Process in a number of steps
  - Binding of ORC to genome (G2/M phase)
  - Licensing (G1 phase)
  - Initiation (firing)
  - Replication
- Try to develop a rudimentary model for licensing-firing-replication
- Treat CDK fluctuation as external signal



Dynamics

- Discrete
  - Firing of origins
  - Destruction of origins by replication
- Continuous
  - Movement of replication along genome
  - Speed depends on location along genome
- Stochastic
  - Location of origins
  - Firing of origins
    - Early vs. late
    - Weak vs. strong

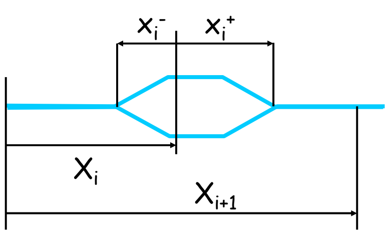
Model data

- Genome length N, normalized to $x \in [0, 1]$
- # of origins of replication (n)
- p(x) p.d.f. of origin positions on genome
- $\lambda(x)$ firing rate of origin at position x
- v(x) forking speed at position x

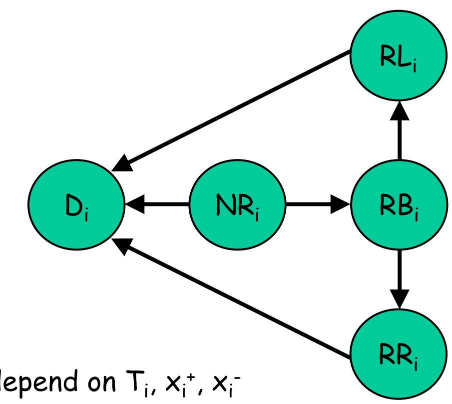
Stochastic terms

- Extract origin positions $X_i \sim p(x),\ i = 1, \ldots, n$
- Extract firing time, T_i, of origin i

$$P\{T_i > t\} = e^{-\lambda(X_i)t}$$



Discrete dynamics (origin i)



Guards depend on T_i, x_i^+, x_i^-

Continuous dynamics (origin i)

- Progress of forking process

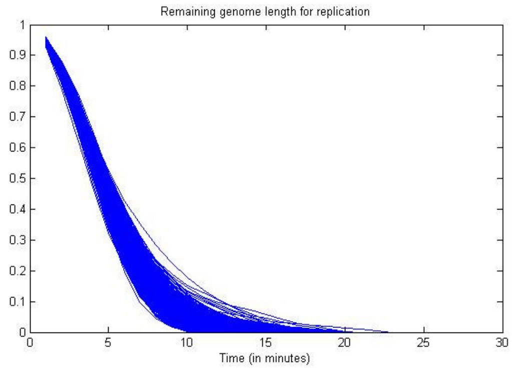
$$\dot{x}_i^+ = \begin{cases} v(X_i + x_i^+) & \text{if } q(i) \in \{RB, RR\} \\ 0 & \text{otherwise} \end{cases}$$

$$\dot{x}_i^- = \begin{cases} v(X_i - x_i^-) & \text{if } q(i) \in \{RB, RL\} \\ 0 & \text{otherwise} \end{cases}$$

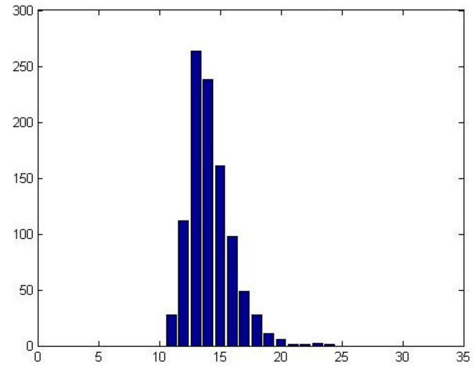
Simulation

- Model a PDP (a simple one!)
- Simulate for *Schizzosacharomyces pombe*
  - N = 12 Mbases
  - n = 400
  - v(x) = 3kbases/minute
- Simple case
  - p(x) uniform
  - $\lambda(x)$ = 0 or infinity

Monte-Carlo simulation



MC estimate of S-phase duration



Research issues

- Validation of experimental data (e.g. average S-phase duration)
- Suggest omissions and new experiments
- Promote understanding, e.g.
  - What determines origin positioning?
  - Is there an advantage to stochastics?
  - Why do some organisms prefer deterministic origin positions?
  - Is there early-late vs. weak-strong distinction?

3.2 Air traffic

Features

- Safety Critical

Features

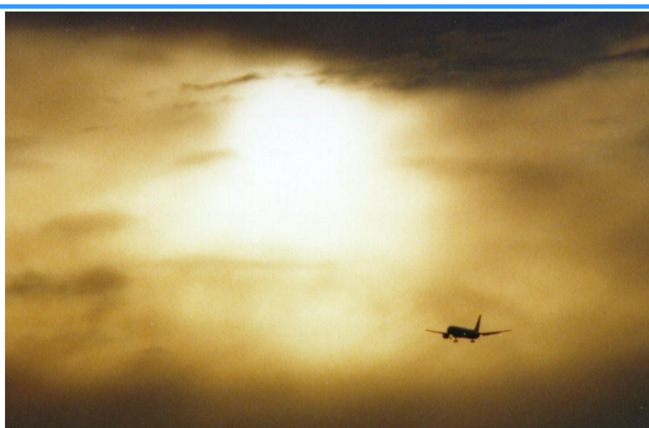
- Safety Critical
- Complex
- Naturally hybrid

Features

- Safety Critical
- Complex
- Naturally hybrid
- Regulations and procedures
- Critical infrastructure
- Human factors
- Remarkably it works …
- … but demand is increasing!

Technological considerations

Socio-economic considerations

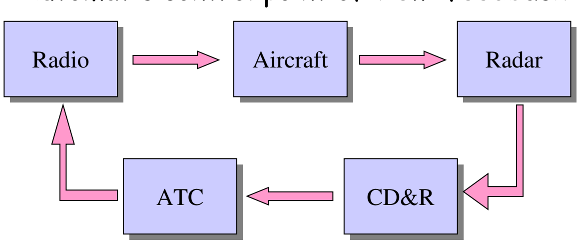


Scope for Improvement

- Naturally distributed system
- Centralised by operational constraints
- Due to
  - Historical reasons
  - Limitation of human operators
- ATM research aims to ease the latter

Framework

- Bottleneck: safety
- In particular in-flight loss of separation
- Conflict Detection and Resolution CD&R
- Automatic control point of view: feedback



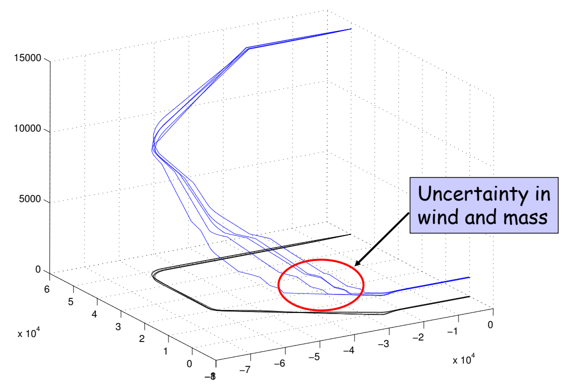
CD&R

- Typically three step process
  - Predict the future
  - Determine whether situation is safety critical
  - Propose resolution manoeuvre
- Last two steps can be merged
- Dynamical model used in all three stages
  - For trajectory prediction
  - Monte-Carlo (particle filter based) conflict detection
  - Monte-Carlo based randomized optimization for conflict resolution

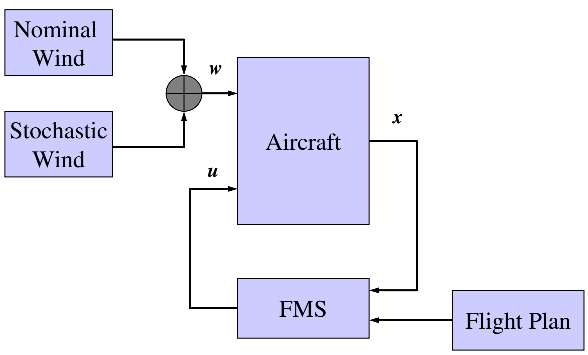
Role of Uncertainty

- CD&R involve predicting the future
- Uncertainty crucial in prediction accuracy
- Probabilistic (stochastic) uncertainty
  - Deviation of weather from nominal
  - Situational awareness errors
  - Faults, malfunctions
- "Deterministic" uncertainty
  - Aircraft mass
  - Thrust settings

Uncertainty in approach



Prediction Model



Aircraft Dynamics

- Flight plan
  - 4D waypoints from CFMU
- Aircraft dynamics:
  - Inspired by BADA
  - Point mass model with extras
  - Parameters from BADA database
- FMS
  - Continuous bank angle controller for tracking and turning
  - Discrete modes for thrust and flight path angle control (acceleration and ROCD)
- Nominal wind from RUC database

Stochastic wind perturbations

- Random field $w(t,P)$
- Correlated in space (P) and time (t)
- Negligible wind in vertical direction
- Correlation in horizontal direction

$$\rho_{xy}(t,t'P,P') = \sigma(z)\sigma(z')e^{-\lambda|t-t'|}e^{-\beta\|(x,y)-(x',y')\|}e^{-\gamma|z-z'|}$$

- Values for parameters: Lincoln labs study
- Strong correlation for horizons of interest
- None of the modelling classes considered above allow this!

Randomized conflict resolution

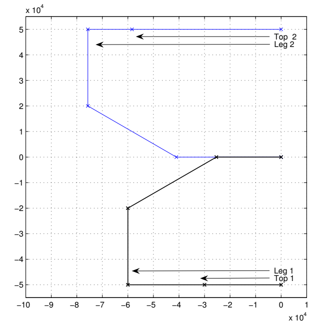
- Probabilistic trajectory prediction models
- Select flight plans to resolve conflicts
- Introduce cost function to select "best" among the safe resolution maneuvers
- Based on advanced Monte-Carlo methods
  - Accommodate complex trajectory prediction models
  - Accommodate general cost criteria
  - Allow recursive and/or parallel implementation

Metropolis-Hastings algorithm

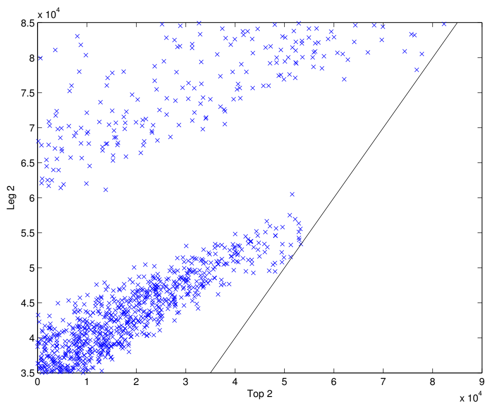
- Related to simulated annealing
- Select cost that reflect
  - Safety
  - Efficiency (deviation from flight plan, flight time, etc.)
- Roughly speaking
  - Select resolution maneuver according to probability distribution
  - Estimate cost by running many simulations
  - "Accept" maneuver with certain probability that depends on whether it is better than previous
  - Probability distribution of "accepted" maneuvers will converge around optimal

Example: Final approach sequencing

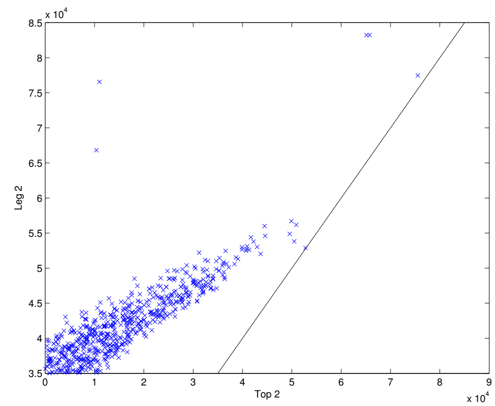
- Two aircraft A1, A2
- Come in at FL160
- A1 flight plan fixed
- Optimize A2 approach
  - Conflict free
  - Reach 1500ft at end
  - Minimum time
- A2 control parameters
  - Top of descent
  - Length of downwind leg
- Probabilistic mass and wind uncertainty



Optimization results

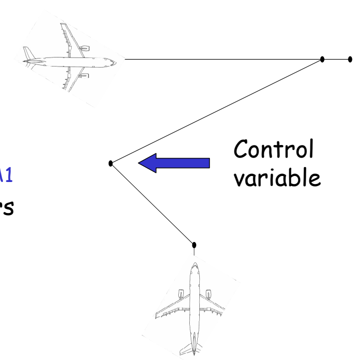


Increased computation

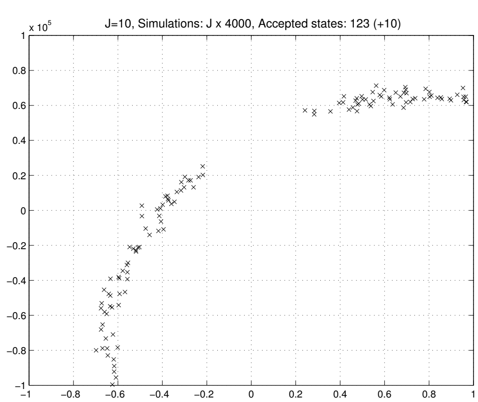


Example: TMA sequencing

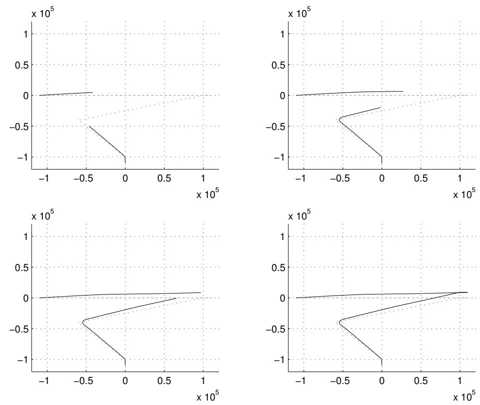
- Two aircraft A1, A2
- Level flight
- A1 flight plan fixed
- Optimize A2 plan
  - Conflict free
  - Exit 5 minutes after A1
- A2 control parameters
  - Position of middle way point
- Probabilistic wind uncertainty



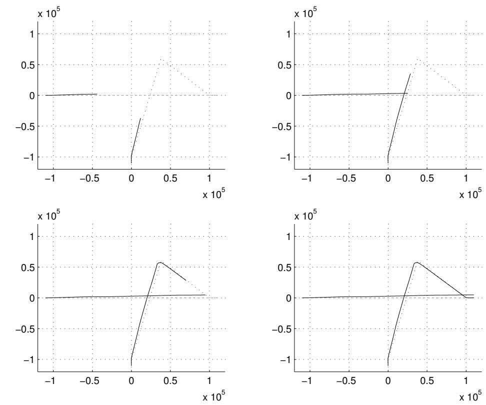
Optimization results



Non-crossing safe maneuvers



Crossing safe maneuvers



Concluding remarks

- SHS different mixtures of discrete, continuous and probabilistic terms
- Modeling of SHS rich and varied
- Foundation for
  - Theoretical contributions (e.g. optimal control, reachability, stability)
  - Practical implications (e.g. Monte-Carlo simulation)
- Technically very challenging
- Motivated by applications
- Applications even more general
  - E.g. spatio-temporal correlation of wind in ATM

References

1. A. Bensoussan and J.L. Menaldi, "Stochastic hybrid control", *Journal of Mathematical Analysis and Applications*, 249:261-288, 2000.
2. M. Bujorianu, "Extended stochastic hybrid systems and their reachability problem", in R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control*, number 2993 in LNCS, pages 234-249. Springer-Verlag, 2004.
3. M.H.A. Davis, "Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models", *Journal of the Royal Statistical Society, B*, 46(3):353-388, 1984.
4. M.H.A. Davis, *Markov Processes and Optimization*, Chapman & Hall, London, 1993.
5. M.K. Ghosh, A. Arapostathis, and S.I. Marcus, "Optimal control of switching diffusions with application to flexible manufacturing systems", *SIAM Journal on Control Optimization*, 31(5):1183-1204, September 1993.
6. M.K. Ghosh, A. Arapostathis, and S.I. Marcus, "Ergodic control of switching diffusions", *SIAM Journal on Control Optimization*, 35(6):1952-1988, November 1997.
7. J. Hespanha. "Stochastic hybrid systems: Application to communication networks", in R. Alur and G.J. Pappas, editors, *Hybrid Systems: Computation and Control*, number 2993 in LNCS, pages 387-401. Springer-Verlag, 2004.

References

8. J. Hu, J. Lygeros, and S.S. Sastry, "Towards a theory of stochastic hybrid systems", in Nancy Lynch and Bruce~H. Krogh, editors, *Hybrid Systems: Computation and Control*, number 1790 in {LNCS}, pages 160-173, Springer-Verlag, Berlin, 2000.
9. X. Mao, "Stability of stochastic differential equations with Markovian switching", *Stochastic Processes and Applications*, 79:45-67, 1999.
10. J.L. Menaldi, "Stochastic hybrid optimal control models", *Aportaciones Matematicas*, 16:205-250, 2001.
11. C. Yuan and X. Mao, "Asymptotic stability in distribution of stochastic differential equations with Markovian switching", *Stochastic Processes and Applications*, 103:277-291, 2003.
12. G. Pola, M.L. Bujorianu, J. Lygeros and M. Di Benedetto, "Stochastic Hybrid Models: An Overview with Applications to Air Traffic Management", in proceedings of *IFAC Conference on Analysis and Design of Hybrid Systems (ADHS03),* Saint Malo, France, June 16-18, 2003
13. J. Lygeros, K.H. Johansson, S.N. Simic, J. Zhang and S.S. Sastry, Dynamical Properties of Hybrid Automata, IEEE Transactions on Automatic Control, 48(1):2-17, January 2003
14. M. Bujorianu and J. Lygeros, "General stochastic hybrid systems: Modelling and optimal control", in proceedings of *IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004.