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Hybrid Optimal Control

Benedetto Piccoli

CNR - Roma, Italy

b.piccoli@iac.rm.cnr.it

Hybrid systems combine continuous dynamics (differential or difference equations) typical of physical plants and discrete dynamics (automata and logical conditions) typical of control logic. By combining disciplines of computer science and systems and control theory, research on hybrid systems provide a solid theory and computational tools for the analysis, simulation, verification, and control design of "embedded systems", and are used in a large variety of applications (automotive systems, air traffic management, biological systems, process industries, and many others).

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Hybrid optimal control

Benedetto Piccoli

Istituto per le Applicazioni del Calcolo “Mauro Picone” – CNR – Roma

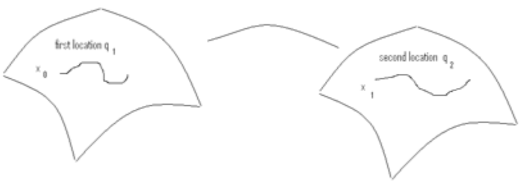
Joint work with A. Amadori, C. D’Apice,
M. Garavello, R. Manzo

Outline of the talk

1. Definition of Hybrid control system
2. Sketch of classical PMP
3. Hybrid Maximum Principle
4. Example
5. Numerical methods via HMP
6. Extensions etc.

1. Definition of HCS

$$\begin{cases} q(t) \equiv q_1 \\ \dot{x}(t) = f_{q_1}(x(t), u_1(t)), & x(t_0) = x_0 \\ \dot{\tau}(t) = 1, & \tau(t_0) = 0 \end{cases}$$



$$\begin{cases} q(t) \equiv q_2 \\ \dot{x}(t) = f_{q_2}(x(t), u_2(t)), & x(t_1) = x_1 \\ \dot{\tau}(t) = 1, & \tau(t_1) = 0. \end{cases}$$

1. Definition of HCS

DEFINITION 2.1. A *hybrid control system* is a 7-tuple $\Sigma = (\mathcal{Q}, M, U, f, \mathcal{U}, J, \mathcal{S})$ such that

H1. $\mathcal{Q}$ is a finite set;
H2. $M = \{M_q\}_{q \in \mathcal{Q}}$ is a family of smooth manifolds, indexed by $\mathcal{Q}$;
H3. $U = \{U_q\}_{q \in \mathcal{Q}}$ is a family of sets;
H4. $f = \{f_q\}_{q \in \mathcal{Q}}$ is a family of maps $f_q : M_q \times U_q \mapsto TM_q$ (TM_q is the tangent bundle of M_q), such that $f_q(x, u) \in T_x M_q$ for every $(x, u) \in M_q \times U_q$;
H5. $\mathcal{U} = \{\mathcal{U}_q\}_{q \in \mathcal{Q}}$ is a family of sets $\mathcal{U}_q$ whose members are maps $u : Dom(u) \to U_q$, defined on some interval $Dom(u) \subset \mathbb{R}$;
H6. $J = \{J_q\}_{q \in \mathcal{Q}}$ is a family of subintervals of $\mathbb{R}^+$;
H7. $\mathcal{S}$ is a subset of $Switch(\Sigma)$, where

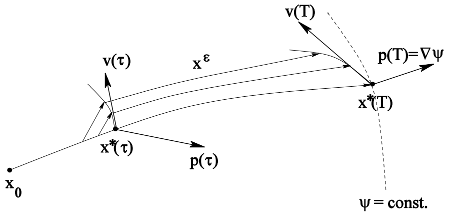
$$Switch(\Sigma) \stackrel{\text{def}}{=} \{(q, x, q', x', u(\cdot), \tau) : q, q' \in \mathcal{Q}, x \in M_q, x' \in M_{q'}, u(\cdot) \in \mathcal{U}_{q'}, \tau \in J_{q'}\}.$$

2. Pontryagin Maximum Principle

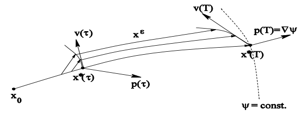
$$\max_{u \in \mathcal{U}} \psi(x(T, u))$$

ons

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = \bar{x},$$



2. Pontryagin Maximum Principle



$$\dot{v}(t) = D_x f(t, x^*(t), u^*(t)) \cdot v(t) \qquad \dot{p}(t) = -p(t) \cdot D_x f(t, x^*(t), u^*(t))$$

$$\frac{d}{dt}(p(t) \cdot v(t)) \equiv 0$$

$$x_\varepsilon(\tau) = x^*(\tau) + \int_{\tau-\varepsilon}^{\tau} [f(t, x_\varepsilon(t), \omega) - f(t, x^*(t), u^*(t))] \, dt$$

$$p(\tau) \cdot \Big[f(\tau, x^*(\tau), \omega) - f(\tau, x^*(\tau), u^*(\tau)) \Big] = p(\tau) \cdot v(\tau) = p(T) \cdot v(T) \le 0$$

2. Pontryagin Maximum Principle

Theorem 1.1 (PMP, free terminal point).

Call $p : [0,T] \mapsto \mathbb{R}^n$ *the solution of the adjoint linear equation*

$$\dot{p}(t) = -p(t) \cdot D_x f(t, x^*(t), u^*(t)), \qquad p(T) = \nabla\psi(x^*(T)).$$

Then the maximality condition

$$p(t) \cdot f(t, x^*(t), u^*(t)) = \max_{\omega \in U} p(t) \cdot f(t, x^*(t), \omega),$$

holds for almost every time $t \in [0,T]$.

3. Hybrid Maximum Principle

- $x_i(\cdot) := x|_{]t_{i-1},t_i[}(\cdot)$ *is an absolutely continuous function in* $]t_{i-1}, t_i[$, *continuously prolongable to* $[t_{i-1}, t_i]$;
- $\frac{d}{dt}x_i(t) = f_{q_i}(x_i(t), u_i(t))$ *for a.e.* $t \in]t_{i-1}, t_i[$;
- $(x_i(t_i), x_{i+1}(t_i)) \in \mathcal{S}_{q_i,q_{i+1}}$ *if* $i = 1, \dots, \nu - 1$;
- $u_{i+1} \in \mathcal{U}_{q_i, x_i(t_i), q_{i+1}, x_{i+1}(t_i)}$ *if* $i = 1, \dots, \nu - 1$.

$$C(\mathbf{X}) = \sum_{j=1}^{\nu} \int_{t_{j-1}}^{t_j} L_{q_j}(x_j(t), u_j(t))\, dt + \sum_{j=1}^{\nu-1} \Phi_{q_j,q_{j+1}}(x_j(t_j), x_{j+1}(t_j)) +$$
$$+\varphi_{q_1,q_\nu}(x_1(t_0), x_\nu(t_\nu)),$$

3 Hybrid Maximum Principle

1. Adjoint covectors: one for each location.

$$\dot{\psi}_i(t) = -< \psi_i(t), \frac{\partial}{\partial x} f_{q_i}(x_i(t), u_i(t)) > -\psi_0 \frac{\partial}{\partial x} L_{q_i}(x_i(t), u_i(t))$$

2. Switching conditions.

$$(-\psi_i(t_i), \psi_{i+1}(t_i)) - \psi_0 \nabla\Phi_{q_i,q_{i+1}}(x_i(t_i), x_{i+1}(t_i)) \in K_i^{\perp}$$

3. Maximized Hamiltonian values are preserved.

$$H_i := \sup\{< \psi_i(t), f_{q_i}(x_i(t), u) > -\psi_0 L_{q_i}(x_i(t), u) : u \in U_{q_i}\},$$

- if $t_j - t_{j-1} \in \mathrm{Int}(J_{q_j})$, then $H_j = H_\nu = 0$;

4. Example: car with gears

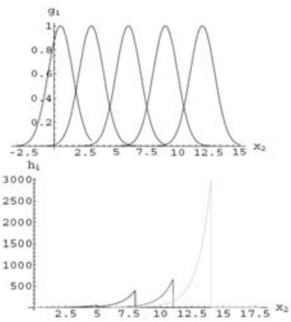
$$\mathcal{Q} = \{A, B\} \times \{1, 2, ..., n\}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = u\, g_i(z_2) \end{cases}$$

$$U_q = [0, 1]$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = u\, h_i(z_2) \end{cases}$$

$$U_q = [-1, 0]$$



4. Example: car with gears

Minimum time problems for fixed final position.

As first example, we consider a minimum time problem from the origin, that is for initial position $z_1 = 0$ and initial velocity $z_2 = 0$, to a fixed final position $z_1 = M$ (no condition on final velocity). This amounts to consider the Lagrangian L such that $L_q \equiv 1$ for any $q \in \mathcal{Q}$ and $c \equiv 0$.

- *the control in each location is always equal to 1;*
- *if* t_i *is the switching time between* (A, i) *and* $(A, i+1)$ $(i \in \{1, \dots, \nu - 1\})$, *then we have that*

$$g_i(x_i^{(2)}(t_i)) = g_{i+1}(x_{i+1}^{(2)}(t_i)).$$

when $x_i^{(2)}$ *and* $x_{i+1}^{(2)}$ *stand for the second scalar components of* x_i *and* x_{i+1} *respectively.*

5. Numerics via HMP

$$\dot{x} = \varphi(x) + \mathcal{B}(x)\, u, \qquad x(0) = x_o. \tag{3.1}$$

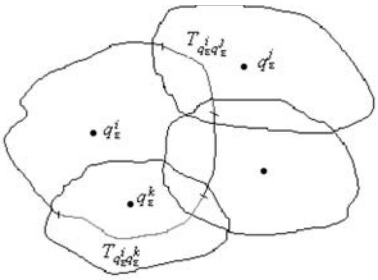
$$\min_u \left\{ \int_0^\infty \frac{1}{2} \left(x^T(t)\, Q\, x(t) + u^T R\, u \right) dt \right\}. \tag{3.2}$$

Definition 3.1 *An Hybridization for the optimal control problem (3.1)-(3.2) is a 4-tuple* $(\varepsilon, \mathbf{Q}, K, T)$, *where* $\varepsilon > 0$, $\mathbf{Q}(\varepsilon) := (q_\varepsilon^j)_{j \in \mathbb{N}}$ *a lattice in* $\mathbb{R}^d$ *with mesh size* ε, $K := (K_\epsilon^j)_{j \in \mathbb{N}}$ *is a collection of cells* K_ε^j *with regular boundaries, containing a neighborhood of* q_ε^j *and* $T = \left(T_{q_\varepsilon^j q_\varepsilon^k}\right)_{j \in \mathbb{N}, k \in \mathbb{N}}$ *are sets with the following properties. The sets* K_ϵ^j *cover* $\mathbb{R}^d$ *and intersect only on zero measure sets,* $T_{q_\varepsilon^j q_\varepsilon^k} \subset K_\epsilon^j$, $\partial K_\varepsilon^j = \cup_k T_{q_\varepsilon^j q_\varepsilon^k}$ *and for every* j, k *and* k' *the sets* $T_{q_\varepsilon^j q_\varepsilon^k}$, $T_{q_\varepsilon^j q_\varepsilon^{k'}}$ *intersect only on a set of zero* $(d-1)$-*dimensional Hausdorff measure.*

5. Numerics via HMP

(C1) $K^j_\epsilon \cap K^k_\epsilon \neq \emptyset$ if and only if $q^k_\epsilon \in V(q^j_\epsilon)$;

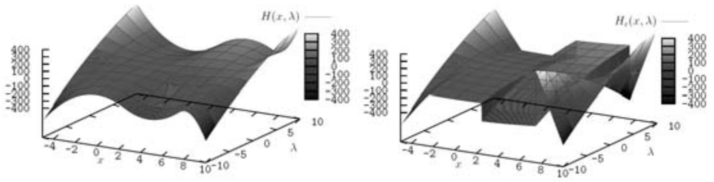
(C2) $\inf_{j,k} d(T_{q^j_\varepsilon q^k_\varepsilon}, \partial K^k_\varepsilon) > \delta > 0$.



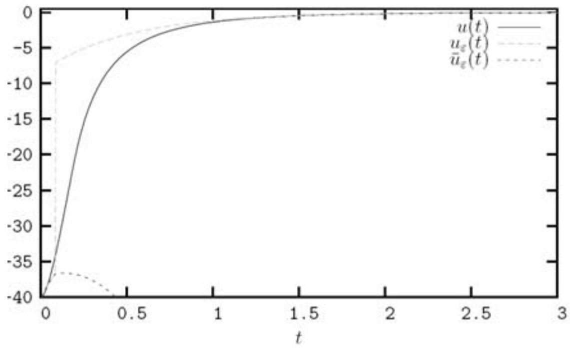
5. Numerics via HMP

$F(x,u) = \varphi(x) + \sqrt{2}\,u.$ $\qquad L(x,u) = x^2 + u^2/2.$

$$\varphi(x) = \begin{cases} 10 + 10(x-8)^2, & \text{if } \ x > 15/2, \\ 25 - 2(x-5)^2, & \text{if } \ 5/2 < x < 15/2, \\ 2x^2, & \text{if } \ x < 5/2, \end{cases}$$



5. Numerics via HMP



6. Hybrid Necessary Principle

Assumption (H). For every fixed $q, q' \in \mathcal{Q}$, $z \in M_q$, $z' \in M_{q'}$, we have $\mathcal{U}_{q,z,q',z'} = \mathcal{U}_{q'}$.

1. If (H) does not hold, then needle variations are not prolongable after location switchings.
2. New admissible variations must be introduced, varying controls after location switchings.
3. Necessary conditions are no more expressed in term of a maximization condition.

6. Open problems

1. Use HMP sistematically in various applications
2. Apply HNP with specific variations
3. Improve numerical schemes

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