1st HYCON PhD School on Hybrid Systems

www.unisi.it

www.ist-hycon.org

Stability and Stabilization of Hybrid Systems

Mikael Johansson

KTH Stockholm, Sweden

mikaelj@s3.kth.se

HYSCOM

IEEE CSS Technical Committee on Hybrid Systems

Information Society Technologies

Siena, July 19-22, 2005 - Rectorate of the University of Siena

Stability and stabilization of hybrid systems

Mikael Johansson
Department of Signals, Sensors and Systems
KTH, Stockholm, Sweden

Goals and class structure

Goal: After these lectures, you should

- Have an overview of some key results on stability and stabilization of hybrid systems
- Be familiar with the computational methods for piecewise linear systems
- Understand how the tools can be applied to (relatively) practical systems

Three lectures:

1. Stability theory
2. Computational tools for piecewise linear systems
3. Applications

Part I – Stability theory

Outline:

- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

Acknowledgements: M. Heemels, ESI

A hybrid systems model

We consider hybrid systems on the form

$$\dot{x}(t) = f(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

where

$x(t) \subset \mathbb{R}^n$ is the continuous state vector
$i(t) \in \{1, 2, \ldots, M\}$ is the discrete state

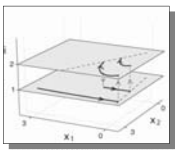
The discrete state indexes vector fields $f(x,i) = f_i(x)$ while $\nu(x,i)$ is the (discontinuous) transition function describing the evolution of the discrete state.

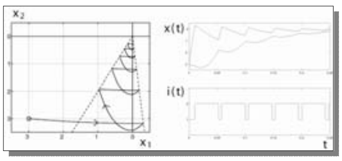
Unless stated otherwise, we will assume that $i(t)$ is piecewise continuous (i.e., that there is only a finite number of mode changes per unit time interval).

For now, disregard issues with sliding modes, zeno, … (precise statements in refs)

Example: a switched linear system

$$\dot{x}(t) = A_{i(t)}x(t)$$
$$i(t^+) = \begin{cases} 2 & \text{if } i(t) = 1 \text{ and } x_2 = -10x_1 \\ 1 & \text{if } i(t) = 2 \text{ and } x_2 = 2x_1 \end{cases}$$





(numerical values for the matrices A_i can be found in the notes for Lecture 2)

Stability concepts

Focus: stability of equilibrium point (in the continuous state-space) $x = 0$

Global asymptotic stability (GAS): ensure that

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for all initial states } (x(0), i(0))$$

Global uniform asymptotic stability (GUAS): ensure that

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for all initial states } (x(0), i(0)) \text{ and for all piecewise continuous } i(t)$$

(i.e., uniformly in $i(t)$)

Three fundamental problems

Problem P1: Under what conditions is

$$\dot{x}(t) = f(x(t), i(t))$$

GAS for all (piecewise continuous) switching signals $i(t)$?

Problem P2: Given vector fields $f(x,i) = f_i(x)$, design switching strategy $\nu(x,i)$:

$$\dot{x}(t) = f(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

is globally asymptotically stable.

Problem P3: determine if a given switched system

$$\dot{x}(t) = f(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

is globally asymptotically stable.

Part I – Stability theory

Outline:

- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

Aim: establishing common grounds by reviewing fundamentals.

Lyapunov theory for smooth systems

Theorem. *Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function such that*

(i) $V(x) \to \infty$ *as* $\|x\| \to \infty$	*(radially unbounded)*
(ii) $V(0) = 0$ *and* $V(x) > 0$ *if* $x \neq 0$	*(positive definite)*
(iii) $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$ *for all* $x \neq 0$	*(decreasing)*

then $x = 0$ is globally asymptotically stable.

Interpretation: Lyapunov function is an abstract measure of system energy
System energy should decrease along all trajectories.

Converse theorem

Under appropriate technical conditions (mainly smoothness of the vector fields)

Theorem. *If $x = 0$ is a GAS equilibrium of $\dot{x} = f(x)$, then there exists a radially unbounded Lyapunov function $V(x)$*

Consequence: worthwhile to search for Lyapunov functions (but how?)

Stability of linear systems

Theorem. *The following statements are equivalent:*

(i) The linear system $\dot{x} = Ax$ is asymptotically stable

(ii) There is a quadratic Lyapunov function

$$V(x) = x^T P x$$

for some positive definite matrix $P > 0$ such that

$$A^T P + PA < 0$$

Moreover, for every asymptotically stable A and for any $Q > 0$ there is a $P > 0$ such that the following *Lyapunov equality* holds

$$A^T P + PA = -Q$$

Partial proof

(ii)→(i): Assume that there is $P > 0$ such that $A^T P + PA < 0$. Then there exists an $\epsilon > 0$ such that

$$A^T P + PA + \epsilon P < 0$$

Letting $V(x) = x^T P x$, then for all $t \in \mathbb{R}$

$$\frac{d}{dt} V(x(t)) + \epsilon V(x(t)) = x^T(t)(A^T P + PA)x(t) + \epsilon x^T(t) P x(t)$$
$$= x^T(t)(A^T P + PA + \epsilon P)x(t) \leq 0$$

After integration, this yields for all $t \leq t_0$.

$$x^T(t) P x(t) \leq x^T(t_0) P x(t_0) e^{-\epsilon t}$$

Now use that $\lambda_{min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{max}(P)\|x\|^2$ to infer

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 \frac{\lambda_{max}(P)}{\lambda_{min}(P)} e^{-\epsilon t}$$

Stability of discrete-time systems

Theorem. *Let $x=0$ be an equilibrium point of $x(t_{k+1}) = f(x(t_k))$, and let $V: \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function s.t.*

(i) $V(x) \to \infty$ as $\|x\| \to \infty$
(ii) $V(0) = 0$ *and* $V(x) > 0$ *if* $x \neq 0$
(iii) $\Delta V(x) = V(f(x(t_k))) - V(x(t_k)) < 0$ *for all* $x \neq 0$

then $x=0$ is globally asymptotically stable.

Interpretation: System energy should decrease at every sampling instant (event)

Performance analysis

Lyapunov-like techniques are also useful for estimating system performance.

Theorem. *If there exists a radially unbounded, positive definite storage function $V(x)$ satisfying*

$$\frac{\partial V(x)}{\partial x} f(x, w) \leq \gamma^2 \|w\|^2 - \|y\|^2 \qquad \forall x, w$$

then the smooth nonlinear system

$$\dot{x}(t) = f(x(t), w(t))$$
$$y(t) = g(x(t))$$

has L_2-gain less than γ (i.e., $\int_0^t \|y(s)\|^2 \, ds \leq \gamma^2 \int_0^t \|w(s)\|^2 \, ds \ \forall t$)

Part I – Stability theory

Outline:
- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

Content:
- Guaranteeing stability independent of switching strategy
- Design a stabilizing switching strategy
- Prove stability for a given switching strategy

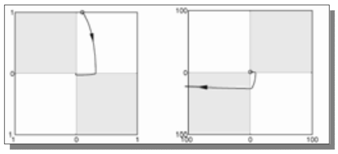
Switching between stable systems

Question: does switching between stable linear dynamics always create stable motions?

Answer: no, not necessarily.

$$\dot{x} = A_{i(x)}x \text{ for } x \in X_i \text{ with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

Both systems are stable, share the same eigenvalues, but stability depends on switching!



P1: Stability for arbitrary switching signals

Problem: when is the switched system

$$\dot{x}(t) = f(x(t), i(t)) = f_{i(t)}(x(t))$$

globally asymptotically stable for all (piecewise continuous) switching signals $i(t)$?

Claim: only if there is a radially unbounded Lyapunov function for each subsystem

$$\dot{x}(t) = f_i(x(t))$$

(can you explain why?)

The common Lyapunov function approach

In fact, if the submodels are smooth, the following results hold.

Theorem. *If all submodels share a common positive definite radially unbounded Lyapunov function, then the switched system is GUAS.*

Theorem. *If the switched system is GUAS, then all submodels share a positive definite radially unbounded common Lyapunov function.*

Hence, common Lyapunov functions necessary and sufficient.

Switched linear systems

For switched linear systems

$$\dot{x}(t) = A_{i(t)}(x(t))$$

it is natural to look for a common quadratic Lyapunov function

$$V(x) = x^T P x \qquad \text{with } P > 0$$

$V(x)$ is a common Lyapunov function if

$$\dot{V}(x) = x^T(A_i^T P + P A_i)x < 0 \text{ for all } i = 1, 2, \ldots, M$$

Common quadratic Lyapunov function found by solving linear matrix inequalities

$$P > 0 \qquad A_i^T P + P A_i < 0 \text{ for all } i = 1, 2, \ldots, M$$

(systems that admit quadratic Lyapunov function are sometimes called *quadratically stable*)

Infeasibility test

It is also possible to prove that there is no common quadratic Lyapunov function:

Theorem. *If there exist positive definite matrices $R_i > 0$ such that*

$$\sum_{i=1}^{M} R_i A_i^T + A_i R_i > 0$$

then there is no $P > 0$ such that

$$A_i^T P + P A_i < 0 \qquad \forall i \in \{1, \ldots, M\}$$

Example

Question: Does GUAS of switched linear system imply existence of a common quadratic Lyapunov function?

Answer: No, the system given by

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

is GUAS, but does not admit any common quadratic Lyapunov function since

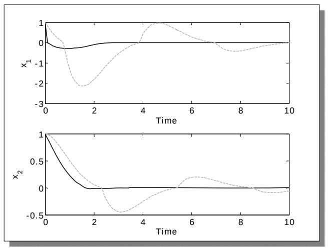
$$R_1 = \begin{pmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{pmatrix}$$

satisfy the infeasibility condition.

(there is, however, a common *piecewise quadratic* Lyapunov function)

Example

Sample trajectories of switched system (under two different switching strategies)



Even if solutions are very different, all possible motions are asymptotically stable

P2: Stabilization

Problem formulation: given matrices A_i, find switching rule $\nu(x,i)$ such that

$$\dot{x}(t) = A_{i(t)}x(t)$$
$$i(t^+) = \nu(x(t), i(t))$$

is asymptotically stable.

Stabilization of switched linear systems

Theorem. *If there exist $\alpha_i > 0$ with $\sum_i \alpha_i = 1$ such that*

$$\dot{x}(t) = \sum_i \alpha_i A_i x(t) := A_{eq} x(t)$$

is globally asymptotically stable, then there exists a switching strategy that makes the switched system globally asymtotically stable.

Note: if only two subsystems, then condition is also necessary.

Stabilizing switching rules (I)

A state-dependent switching strategy can be designed from Lyapunov function for A_{eq}

Solve Lyapunov equality $A_{eq}^T P + P A_{eq} = -Q$. It follows that

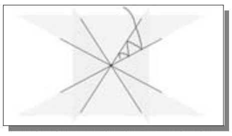
$$\sum_i \alpha_i x^T (A_i^T P + P A_i) x = x^T (A_{eq}^T P + P A_{eq}) x = -x^T Q x < 0$$

Consequence: for each x, at least one mode satisfies $x^T (A_i^T P + P A_i) x(t) < 0$

This implies, in turn, that the switching rule

$$\nu(x) = \arg\min x^T (A_i^T P + P A_i) x$$

is well-defined for all x and that it generates globally asymptotically stable motions.



Stabilizing switching rules (II)

An alternative switching strategy is to activate mode i a fraction α_i of the time, e.g.,

$$i(t^+) = \begin{cases} 1 & \text{if } 0 \le t < \alpha_1 T \\ 2 & \text{if } \alpha_1 T \le t < (\alpha_1 + \alpha_2) T \\ \vdots & \\ N & \text{if } \sum_{i=1}^{N-1} \alpha_i T \le t < T \end{cases}$$

(the strategy repeats after a duty cycle of T seconds). The "average dynamics" is then

$$\dot{x} = A_{eq} x$$

and for sufficiently small T the spectral radius of

$$\exp(A_1 \alpha_1 T) \exp(A_2 \alpha_2 T) \cdots \exp(A_N \alpha_N T)$$

is less than one (i.e., the state at the beginning of each duty cycle will tend to zero)

Example

Consider the two subsystems given by

$$A_1 = \begin{pmatrix} -0.5 & 1 \\ 100 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -100 \\ -0.5 & -1 \end{pmatrix}$$

Both subsystems are unstable, but the matrix $A_{eq}=0.5A_1+0.5A_2$ is stable.

State-dependent switching: set Q=I, solve Lyapunov equation to find

$$P = \begin{pmatrix} 0.5700 & 0.0015 \\ 0.0015 & 0.5728 \end{pmatrix}$$

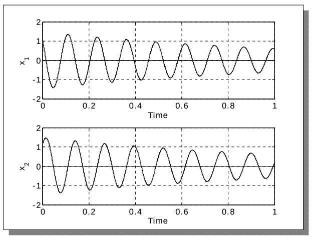
Time-dependent switching: choose duty cycle T such that spectral radius of

$$\exp(A_1 T/2) \exp(A_2 T/2)$$

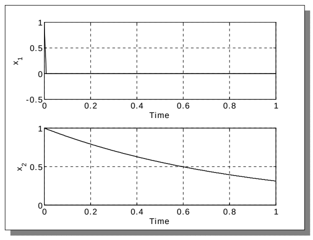
is less than one. Alternate between modes each T/2 seconds.

Example cont'd

Time-driven switching



State-dependent switching



P3: Stability for a given switching strategy

Problem: how can we verify that the switched system

$$\dot{x}(t) = f(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

is globally asymptotically stable?

Stability for given switching strategy

For simplicity, consider a system with two modes, and assume that

$$\dot{x}(t) = f_i(x(t)) \qquad i = 1,\ 2$$

are globally asymptotically stable with Lyapunov functions V_i

Even if there is no common Lyapunov function, stability follows if

$$V_{i(t_{k-1})}(x(t_k)) = V_{i(t_k)}(x(t_k)) \qquad \forall k = 1, 2, \ldots$$

where t_k denote the switching times.

Reason: V_i is a continuous Lyapunov function for the switched system.

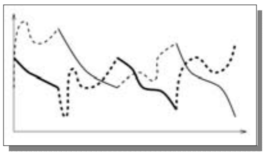
Multiple Lyapunov function approach

Theorem. *Consider the switched system where all submodels $\dot{x} = f_i(x)$ are globally asymptotically stable with Lyapunov functions V_i.*

Suppose that for each pair of switching times (t_k, t_l), $k < l$ with $i(t_k) = i(t_l) = \hat{i}$ and $i(t_m) \neq \hat{i}$ for $t_k < t_m < t_l$, we have

$$V_i(x(t_l)) \leq V_i(x(t_k)) - \rho(|x(t_k)|)$$

then the switched system is globally asymptotically stable.



Note: need to know switching times → very hard to apply (more later).

Multiple Lyapunov function approach

Weaker versions exist:

– No need to require that submodels are stable, sufficient to require that all submodels admit *Lyapunov-like* functions:

$$V_i(x) > 0 \qquad \text{for } x \in X_i$$
$$\frac{\partial V_i(x)}{\partial x} f_i(x) < 0 \qquad \text{for } x \subset X_i$$

where X_i contains all x for which submodel f_i can be activated.

– Can weaken the condition that V_i should decrease along trajectories of $f_i(x)$

See the references for details and precise statements.

Summary

A whirlwind tour:
- *selected* results on stability and stabilization of hybrid systems

Three specific problems
- Guaranteeing stability independent of switching signal
- Design a stabilizing switching strategy (stabilizability)
- Prove stability for a given switching strategy

Focus has been on Lyapunov-function techniques
- Alternative approaches exist!

Strong theoretical results, but hard to apply in practice
- Can be overcome by developing automated numerical techniques (Lecture 2!)

References

R. A. DeCarlo, M. S. Branicky, S. Pettersson and B. Lennartsson, “Perspectives and results on the stability and stabilizability of hybrid systems”, Proceedings of the IEEE, Vol. 88, No. 7, July 2000.

J. P. Hespanha, “Stabilization through hybrid control”, UNESCO Encyclopedia of Life Support Systems”, 2005.

M. Johansson, “Piecewise linear control systems – a compuational approach”, Springer Lecture Notes in Control and Information Sciences no. 284, 2002.

J. Goncalves, ”Constructive Global Analysis of Hybrid Systems”, Ph.D. Thesis, Massachusetts Institute of Technology, September 2000.

Stability and stabilization of hybrid systems

Mikael Johansson
Department of Signals, Sensors and Systems
KTH, Stockholm, Sweden

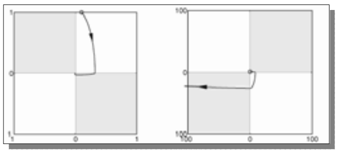
Switching between stable systems

Question: does switching between stable linear dynamics always create stable motions?

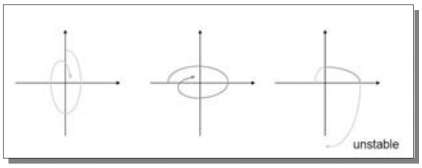
Answer: no, not necessarily.

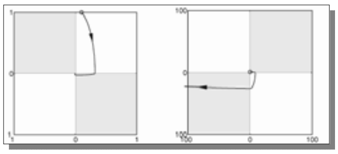
$$\dot{x} = A_{i(x)}x \text{ for } x \in X_i \text{ with } A_1 = \begin{pmatrix} -0.1 & 1 \\ -10 & -0.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -0.1 & 10 \\ -1 & -0.1 \end{pmatrix}$$

Both systems are stable, share the same eigenvalues, but stability depends on switching!



Switching between stable systems



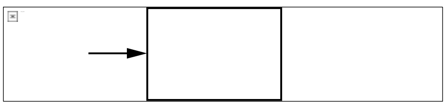


Part II – Computational tools

- Piecewise linear systems
- Well-posedness and solution concepts
- Linear matrix inequalities
- Piecewise quadratic stability
- Extensions

Computational stability analysis: philosophy

Aim: develop analysis tools that
- are computationally efficient (e.g. run in polynomial time)
- work for *most* practical problem instances
- produce guaranteed results (when they work)



Piecewise linear systems

Piecewise linear system:

1. a subdivision of $\mathbb{R}^n$ into regions X_i

$$\bigcup_{i=1}^{M} X_i \subseteq \mathbb{R}^n$$

we will assume that X_i are polyhedral and disjoint (only share common boundaries)

2. (possibly different) affine dynamics in each region

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

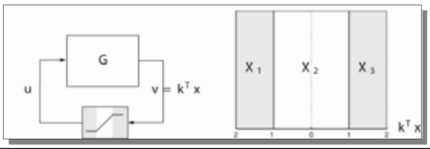
Example

Saturated linear system: $\dot{x} = Ax + b\,\mathrm{sat}(v),\ v = k^T x$

Three disjoint regions: negative saturation, linear operation, and positive saturation

$$\dot{x} = \begin{cases} Ax - b & x \in X_1 \\ (A - bk^T)x & x \in X_2 \\ Ax + b & x \in X_3 \end{cases}$$

Cells are polyhedral (i.e., can be described by a set of linear inequalities)

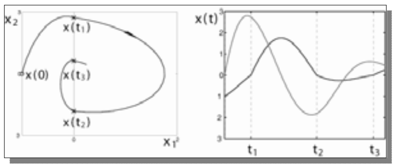


Well-posedness and solutions

Definition. *Let $x(t) \in \cup_{i \in I} X_i$ be an absolutely continuous function. We say that $x(t)$ is a trajectory of the system*

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

on $[t_0, t_f]$ if, for almost all $t \in [t_0, t_f]$, the equation $\dot{x}(t) = A_i x(t) + a_i + B_i u(t)$ holds for all i with $x(t) \in X_i$.

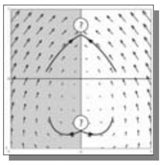


Trajectories: existence and uniqueness

Observation: trajectories may not be unique, or may not exist.

Example:

$$\begin{cases} \dot{x}_1 = -2x_1 - 2x_2 \mathrm{sgn}(x_1) \\ \dot{x}_2 = x_2 + 4x_1 \mathrm{sgn}(x_1) \end{cases}$$



Initial values in $S_1^- = \{x \mid x_1 = 0 \wedge x_2 \leq 0\}$ create non-unique trajectories.

Trajectories that reach $S_1^+ = \{x \mid x_1 = 0 \wedge x_2 \geq 0\}$ cannot be continued

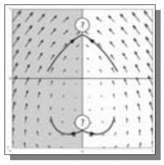
Attractive sliding modes

Would like to single out situations with non-existence of solutions.

Definition. *The system*

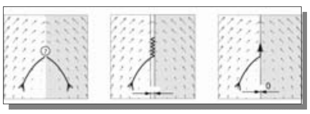
$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

is said to have an attractive sliding mode at x_s if there exists a trajectory with final state x_s but no trajectory with initial state x_s.



Generalized solutions

Solution concepts for sliding modes typically averages dynamics in neighboring regions.



Definition. *Let $x(t) \in \cup_{i \in I} X_i$ be an absolutely continuous function. We say that $x(t)$ is a Filippov solution of (1) on $[t_0, t_f]$ if*

$$\dot{x}(t) \in \overline{\mathrm{co}}_{k \in K(t)} \{A_k x(t) + a_k + B_k u(t)\}$$

for almost all t, where K is the set of indicies such that $x(t) \in X_k$.

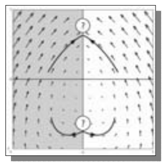
Note: Filippov solutions may remain on cell boundaries, but are not necessarily unique.

Equivalent dynamics on sliding modes

Example: Piecewise linear system

$$\begin{cases} \dot{x}_1 = -2x_1 - 2x_2 \mathrm{sgn}(x_1) \\ \dot{x}_2 = x_2 + 4x_1 \mathrm{sgn}(x_1) \end{cases}$$

on $S_1^+ = \{x \mid x_1 = 0 \wedge x_2 \geq 0\}$



Filippov solution should satisfy $\dot{x}(t) \in \alpha A_1 x(t) + (1 - \alpha) A_2 x(t)$ for some $\alpha \in [0, 1]$

If x(t) should stay on S_1^+, we must have $\dot{x}_1(t) = 0$, i.e.,

$$\alpha \cdot 2x_2 + (1 - \alpha) \cdot (-2x_2) = x_2(4\alpha - 2) = 0$$

The only solution is given by $\alpha = 1/2$, resulting in the unique sliding mode dynamics

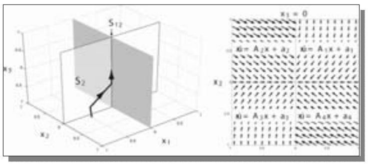
$$\dot{x}_1 = 0, \qquad \dot{x}_2 = x_2$$

Non-uniqueness of sliding dynamics

Observation: sliding mode dynamics on intersecting boundaries often non-unique

Example:

$$\dot{x}_1 = x_2 - \mathrm{sgn}(x_1)$$
$$\dot{x}_2 = x_3 - \mathrm{sgn}(x_2)$$
$$\dot{x}_3 = -2x_1 - 4x_2 - 4x_3 - x_3\mathrm{sgn}(x_2)\mathrm{sgn}(x_1+1)$$



Filippov solutions on the set $\mathcal{S}_{12} = \{x \mid x_1 = 0 \wedge x_2 = 0 \wedge |x_3| \le 1\}$ are not unique. (can you explain why?)

Valid Filippov solutions on $\mathcal{S}_{12}$ have time constant that differ a factor four or more.

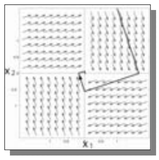
Establishing attractivity of sliding modes

Observation: non-trivial to detect that a pwl system has attractive sliding modes

Example: The piecewise linear system

$$\dot{x}_1 = -\mathrm{sgn}(x_1) + 2\mathrm{sgn}(x_2)$$
$$\dot{x}_2 = -2\mathrm{sgn}(x_1) - \mathrm{sgn}(x_2)$$

has a sliding mode at the origin.



However, determining that it is attractive is not easy
- Vector field considerations or quadratic Lyapunov functions cannot be used (why?)
- Finite-time convergence to the origin can be established by noting that

$$\frac{d}{dt}(|x_1| + |x_2|) = -2$$

Key points

Piecewise linear systems: polyhedral partition and locally affine dynamics

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

For general piecewise linear systems, solution concepts are non-trivial
- Trajectories may not be unique, or may not exist (unless continuous right-hand side)
- Meaningful solution concepts for attractive sliding modes exist (e.g. Filippov solutions)

Introducing “new modes” on cell boundaries with equivalent sliding dynamics is not easy
- Sliding modes may occur on any intersection of cell boundaries
- Hard to determine if potential sliding mode is attractive
- Dynamics of sliding modes may be non-unique and non-linear

Part II – Computational tools

- Piecewise linear systems
- Well-posedness and solution concepts
- Linear matrix inequalities
- Piecewise quadratic stability
- Discrete-time hybrid systems

Linear matrix inequalities

Linear matrix inequality (LMI): An inequality on the form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0$$

where F_i are symmetric matrices, and X>0 denotes that X is positive definite.

Example: The condition $P > 0$ on standard form:

$$p_{11}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + p_{12}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + p_{22}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} > 0$$

LMI features

- Optimization under LMI constraints is a *convex* optimization problem
  - Strong and useful theory, e.g. duality (we have already used it once – when?)
- Multiple LMIs is an LMI
  - Example: Lyapunov inequalities $P > 0,\ A^T P + PA < 0$ equivalent to single LMI

$$\begin{bmatrix}P & 0\\ 0 & -A^T P - PA\end{bmatrix} > 0$$

- Efficient software and convenient user interfaces publicly available
  - Example: YALMIP interface by J. Löfberg at ETHZ
- S-procedure, Shur complements, … and much more!

Example: Quadratic stabilization

Recall from Lecture 1 that $V(x) = x^T P x$ quarantees that

$$\dot{x}(t) = A_{i(t)}(x(t))$$

is GAS for all switching signals i(t) (i.e., GUAS) if there exists P such that

$$P > 0$$
$$A_i^T P + P A_i < 0 \qquad \forall i \in \{1, 2, \ldots, M\}$$

an LMI condition!

Consequence: quadratic Lyapunov function found efficiently (if it exists)!

Quadratic stability of PwL systems

$V(x) = x^T P x$ is a Lyapunov function for the piecewise linear system

$$\dot{x} = A_i x \qquad x \subset X_i$$

if we have

$$x^T P x > 0 \qquad \forall x \neq 0$$
$$x^T (A_i^T P + P A_i) x < 0 \qquad \forall x \in X_i \backslash 0$$

Note: unnecessary to require that $A_i^T P + P A_i < 0$

How can we bring the restricted decreasing conditions into the LMI framework?

S-procedure

When does it hold that, for all x,

$$x^T R x \geq 0 \Rightarrow x^T P x \geq 0$$

(i.e., that non-negativity of quadratic form $x^T R x$ implies non-neagivity of $x^T P x$)

Simple condition: there exists $\tau \in \mathbb{R}_+$ satisfying the LMI $P \geq \tau R$

Extension to multiple quadratic forms: if there exist $\tau_i \geq 0$ such that

$$P - \sum_i \tau_i R_i \geq 0$$

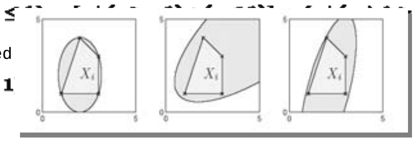
then $(x^T R_1 x \geq 0) \wedge (x^T R_2 x \geq 0) \cdots \Rightarrow x^T P x \geq 0$

(non-trivial fact: the simple condition is necessary if there exists an u: $u^T R u > 0$)

Bounding polyedra by quadratic forms

Example: The polyhedron

$X = \{x \mid |x| \leq \ldots 0) \wedge (1 - x \geq 0)\}$



can be described

$q(x) = \tau(x + 1 \ldots$

for $\tau \geq 0$

In general: for polyhedra $X_i = \{x \mid E_i x + e_i \succeq 0\}$ the quadratic form

$$q(x) = \left([E_i \quad e_i] \begin{bmatrix} x \\ 1 \end{bmatrix} \right)^T U_i \left([E_i \quad e_i] \begin{bmatrix} x \\ 1 \end{bmatrix} \right) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{E}_i^T U_i \bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix}$$

is non-negative for all $x \in X_i$ if W_i has non-negative entries

Quadratic stability cont'd

Consider the piecewise linear system

$$\dot{x} = A_i x \qquad \text{for } x \in X_i = \{x \mid E_i x \succeq 0\}$$

(no affine terms, all regions contain the origin). Then, we can state the following

Theorem. *If there exists a positive definite matrix P and matrices U_i with non-negative entries such that*

$$A_i^T P + P A_i + E_i^T U_i E_i < 0$$

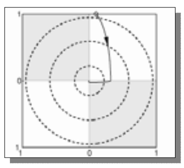
then every Filippov solution tends to zero exponentially.

Example

Recall the switching system $\dot{x} = A_1 x$ for $x_1 x_2 \geq 0$, $\dot{x} = A_2 x$ for $x_1 x_2 \leq 0$ with

$$A_1 = \begin{pmatrix} -0.1 & 1 \\ -10 & -0.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -0.1 & 10 \\ -1 & -0.1 \end{pmatrix}$$

from Lecture 1. Applying the above procedure, we find $P = I$, e.g., $V(x) = x^T x$.



(stability cannot be verified without S-procedure terms – can you explain why?)

Piecewise quadratic Lyapunov functions

Natural to consider continuous, *piecewise quadratic*, Lyapunov functions

$$V(x) = x^T P_i x + 2q_i^T x + r_i = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \text{for } x \in X_i$$

Surprisingly, such functions can also be computed via optimization over LMIs.

Relation to multiple Lyapunov functions:

- Local expressions for V(x) are Lyapunov-like functions for associated dynamics (stronger relationship will emerge in the extensions)

Convenient notation

Use the augmented state vector

$$\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and re-write system dynamics as

$$\begin{bmatrix} \dot{\bar{x}} \\ y \end{bmatrix} = \left[\begin{array}{cc|c} A_i & a_i & B_i \\ 0_{1\times n} & 0 & 0_{1\times m} \\ \hline C_i & c_i & D_i \end{array}\right] \begin{bmatrix} \bar{x} \\ u \end{bmatrix} = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \begin{bmatrix} x \\ 1 \end{bmatrix}$$

When analyzing properties of the equilibrium $x = 0$ we let

$I_0 \subseteq I$ be the set of indices for regions containing origin

$I_1 \subseteq I$ be the set of indices for regions that do not contain origin

and assume that $a_i = c_i = 0$ for $i \in I_0$

Enforcing continuity

How to ensure that the Lyapunov function candidate

$$V(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{x}^T \bar{P}_i \bar{x} \qquad \text{for } x \in X_i$$

is continuous across cell boundaries?

Proposition. $\bar{x}^T \bar{P}_i \bar{x} = \bar{x}^T \bar{P}_j \bar{x}$ *for all* $x \in X_i \cap X_j = \{x \mid \bar{h}_{ij}^T \bar{x} = 0\}$ *if and only if there exists* $\bar{l}_{ij} \in \mathbb{R}^{n+1}$ *such that*

$$\bar{P}_i = \bar{P}_j + \bar{h}_{ij}^T \bar{l}_{ij} + \bar{l}_{ij}^T \bar{h}_{ij}$$

Enforce one linear equality for each cell boundary.

Enforcing continuity (II)

Alternative: direct parameterization (when solver cannot treat equality constraints)

For each region, construct continuity matrices $\bar{F}_i = [F_i \quad f_i]$ such that

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x} \text{ for all } x \in X_i \cap X_j$$

and consider Lyapunov functions on the form

$$V(x) = \bar{x}^T \bar{F}_i^T T \bar{F}_i \bar{x} \text{ for } x \in X_i$$

(the free variables are now collected in the symmetric matrix T)

To make Lyapunov function quadratic in regions that contain origin, we also require

$$f_i = 0 \text{ for } i \in I_0$$

(construction automated in, for example, Pwltools)

Piecewise quadratic stability

Theorem (Piecewise Quadratic Stability). *Consider symmetric matrices T, U_i and W_i such that U_i and W_i have nonnegative entries, while $P_i = F_i^T T F_i$ and $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ satisfy*

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \qquad i \in I_0$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i \end{cases} \qquad i \in I_1$$

Then every trajectory $x(t) \in \cup_{i\in I} X_i$ satisfying

$$\dot{x} = A_i x + a_i \qquad \text{for } x \in X_i$$

tends to zero exponentially.

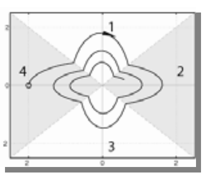
Example

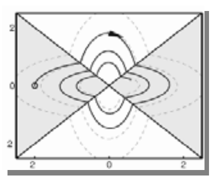
Piecewise linear system with partition shown below,

$$A_1 = A_3 = \begin{bmatrix} -\epsilon & \omega \\ -\alpha\omega & -\epsilon \end{bmatrix}, \qquad A_2 = A_4 = \begin{bmatrix} -\epsilon & \alpha\omega \\ -\omega & -\epsilon \end{bmatrix}.$$

and $\alpha = 5$, $\omega = 1$, $\epsilon = 0.1$

(Clearly) not quadratically stable, but pwQ Lyapunov function readily found.





Potential sources of conservatism

1. Quadratic Lyapunov functions necessary and sufficient for linear systems, *but* piecewise quadratic Lyapunov functions *not necessary* for stability of PWL systems.
2. S-procedure terms $\bar{E}_i^T W_i \bar{E}_i$ are effectively the sum of several quadratic forms
$$\bar{x}^T \bar{E}_i^T W_i \bar{E}_i \bar{x} = \sum_i \sum_j w_{ij} (\bar{e}_i^T \bar{x})^T (\bar{e}_j^T \bar{x})$$
hence, S-procedure is not guaranteed to be loss-less (but better tools exist)
3. Use of affine terms and strict inequalities can also be conservative.

⋮

Extensions

Many extensions possible:

- determining regions of attraction (i.e. non-global stability properties)
- Lyapunov functions that guarantee stability of potential sliding modes
- nonlinear and uncertain dynamics in each region
- performance analysis (e.g. L_2-gains)
- (some) control synthesis
- hybrid systems (overlapping regions) and discontinuous Lyapunov functions
- Lyapunov functionals and Lagrange stability
- stability of limit cycles
- similar tools for discrete-time hybrid systems

⋮

(too much to be covered in this lecture!)

We will sketch a couple of extensions

Performance analysis

Theorem (Upper Bound on L_2 Gain). *Suppose there exist symmetric matrices T, U_i and W_i such that U_i and W_i have non-negative entries, while $P_i = F_i^T T F_i$ and $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ satisfy*

$$0 > \begin{bmatrix} P_i A_i + A_i^T P_i + C_i^T C_i + E_i^T U_i E_i & P_i B_i \\ B_i^T P_i & -\gamma^2 I \end{bmatrix} \quad \text{for } i \in I_0$$

$$0 > \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i^T \bar{P}_i + \bar{C}_i^T \bar{C}_i + \bar{E}_i^T U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i^T \bar{P}_i & -\gamma^2 I \end{bmatrix} \quad \text{for } i \in I_1$$

Then for every trajectory with $x(0) = 0$, $\int_0^\infty (\|x\|_2^2 + \|u\|_2^2)\,dt < \infty$

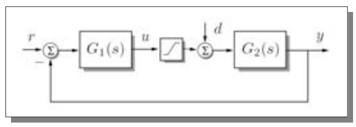
$$\int_0^\infty \|y\|_2^2\,dt \le \gamma^2 \int_0^\infty \|u\|_2^2\,dt.$$

The best upper bound on the L_2 induced gain is achieved by minimizing γ^2 subject to the constraints defined by the inequalities.

Proof. Pre-and postmultiply with (x, u), note that LMIs imply dissipation inequality

Example

Saturated linear system (unit saturation)



$$G_1(s) = \frac{s-3}{16s^2 + s + 2}$$
$$G_2(s) = \frac{s+7}{4s^2 + 3s + 12}$$

Quadratic storage functions fail to bound L_2-gain.

Piecewise quadratic storage function yields bounds

$$5.52 \le \gamma \le 5.54$$

Linear hybrid dynamical systems

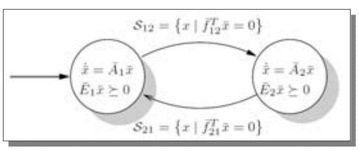
Linear hybrid dynamical system (LHDS)

$$\dot{x}(t) = A_{i(t)} x(t) + a_{i(t)}$$
$$i(t^+) = \nu(x(t), i(t))$$

ν described by finite automaton whose state changes when x hits *transition surfaces*

$$S_{ij} = \{x \mid \bar{f}_{ij}^T \bar{x} = 0\}$$

and for each i, the feasible x can be bounded by a polyhedron $X_i = \{x \mid \bar{E}_i \bar{x} \succeq 0\}$



Discontinuous Lyapunov functions

Multiple quadratic (discontinuous, pwq) Lyapunov function via LMIs

Theorem. *Consider symmetric matrices U_i, W_i with non-negative entries, symmetric matrices P_i, $\bar{P}_i$, and vectors t_{jk}, $\bar{t}_{jk}$ such that*

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \quad i \in I_0 \quad (1)$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i \end{cases} \quad i \in I_1 \quad (2)$$

$$0 < \bar{P}_j - \bar{P}_k + \bar{f}_{jk} \bar{t}_{jk}^T + \bar{t}_{jk} \bar{f}_{jk}^T \quad (j,k) \in T, \quad j \in I_1 \text{ or } k \in I_1 \quad (3)$$

$$0 < P_j - P_k + f_{jk} t_{jk}^T + t_{jk} f_{jk}^T \quad (j,k) \in T, \quad j,k \in I_0 \quad (4)$$

Then every trajectory of the LHDS tends to zero exponentially.

Note: conditions (3,4) imply that V(t) decreases at (potential) points of discontinuity

Example

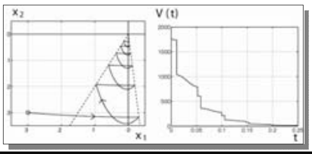
$$\dot{x}(t) = A_{i(t)}x(t)$$

$$i(t^+) = \begin{cases} 2 & \text{if } i(t) = 1 \text{ and } x_2 = -10x_1 \\ 1 & \text{if } i(t) = 2 \text{ and } x_2 = 2x_1 \end{cases}$$

with

$$A_1 = \begin{pmatrix} -1 & -100 \\ 10 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 10 \\ -100 & 1 \end{pmatrix}$$

Trajectories (left) and multiple Lyapunov function found by LMI formulation (right)



Discrete-time versions

Discrete-time piecewise linear systems

$$x[k+1] = A_i x[k] + a_i + B_i u[k] \qquad x[k] \subset X_i$$

and piecewise quadratic Lyapunov (not necessarily continuous) functions

$$V(x[k]) = x[k]^T P_i x[k] + 2q_i^T x[k] + r_i \qquad x[k] \in X_i$$

We have

$$\begin{aligned} \Delta V(x[k]) &= V(x[k+1]) - V(x[k]) \\ &= x[k]^T A_i^T P_j A_i x[k] + 2(a_i^T P_j A_i x + q_j^T A_i) x[k] + a_i^T P_j a_i + 2q_j^T a_i + r_j \\ &\quad - x[k]^T P_i x[k] + 2q_i^T x[k] + r_i \\ &= \begin{bmatrix} x[k] \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P_j A_i - P_i & A_i^T P_j a_i + q_j - q_i \\ (\cdot)^T & a_j^T P_i a_j + 2q_j a_i + r_j - r_i \end{bmatrix} \begin{bmatrix} x[k] \\ 1 \end{bmatrix} \end{aligned}$$

for $x[k] \in X_{ij} = \{x \mid x \in X_i \wedge A_i x + a_i \in X_j\} = \{x \mid \bar{E}_i \bar{x} \succeq 0 \wedge \bar{E}_j \bar{A}_i \bar{x} \succeq 0\}$

Discrete-time versions

Discrete-time globally asymptotically stable if there exist matrices P_i, q_i, r_i, $\underline{U}_{ij}$ where W_{ij} has non-negative entries, and a non-negative scalar ε>0, such that

$$\begin{bmatrix} A_i^T P_j A_i - P_i & A_i^T P_j a_i + q_j - q_i \\ (\cdot)^T & a_j^T P_i a_j + 2q_j a_i + r_j - r_i \end{bmatrix} + \bar{E}_{ij}^T U_{ij} \bar{E}_{ij} \leq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}$$

(note: in most solvers, you will need to treat $X_i,\ i \in I_0$ separately)

Observations:

- Again, LMI conditions, hence efficiently verified!
- Potentially one LMI for every pair (i,j) of modes.

Comparison with alternatives

Biswas *et al.* generated optimal hybrid controllers for randomly generated linear systems, and compared performance of several computational methods

Typical results:

Partitions obtained for 3rd order LTI systems, 2 norm objective						
	50 Stable Systems, $N = 1$			50 Unstable Systems, $N = 1$		
Method	Success	Solution Time	Setup Time	Success	Solution Time	Setup Time
Quadratic	45/50	0.7 sec.	0.4 sec.	43/50	1.1 sec.	0.5 sec.
Piecewise Quadratic	50/50	0.7 sec.	1.3 sec.	50/50	1.9 sec.	2.5 sec.
Common SOS order 4	42/50	7.6 sec.	82.2 sec.	32/50	11.4 sec.	141.9 sec.
Piecewise SOS order 4	35/50	12.1 sec.	100.0 sec.	31/50	80.6 sec.	263.8 sec.

Table 2. The number of regions were between 9 and 15 with 9-47 transitions.

Very strong performance, but computational effort increases rapidly (not shown here)

Summary

Computational tools for stability analysis of a particular class of hybrid systems

Piecewise linear systems

- Partition of state space into polyhedra with locally affine dynamics
- Solution concepts: trajectories and Flippov solutions
- Given a pwl model, it is non-trivial to detect attractive sliding modes

Piecewise quadratic Lyapunov functions

- Efficiently computed via optimization over linear matrix inequalities
- Potentially conservative, but strong practical performance

Many extensions, but much work remains!

References

M. Johansson, "Piecewise linear control systems – a computational approach", Springer Lecture Notes in Control and Information Sciences no 284, 2002.

P. Biswas, P. Grieder, J. Löfberg, M. Morari, "A survey on stability analysis of discrete-time piecewise affine systems", IFAC World Congress, Prague, 2005.

J. Löfberg, YALMIP, http://control.ee.ethz.ch/~joloef/yalmip.msql

S. Hedlund and M. Johansson, "A toolbox for computational analysis of piecewise linear systems", ECC, Karlsruhe, Germany, 2002. (http://www.control.lth.se)

Stability and stabilization of hybrid systems

Mikael Johansson
Department of Signals, Sensors and Systems
KTH, Stockholm, Sweden

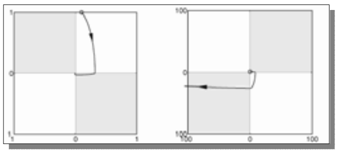
Correction: Switching between stable systems

Question: does switching between stable linear dynamics always create stable motions?

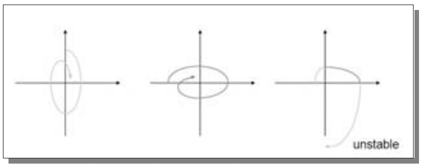
Answer: no, not necessarily.

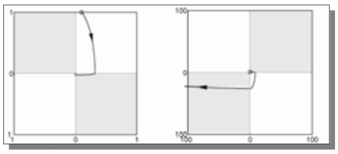
$$\dot{x} = A_{i(x)}x \text{ for } x \in X_i \text{ with } A_1 = \begin{pmatrix} -0.1 & 1 \\ -10 & -0.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -0.1 & 10 \\ -1 & -0.1 \end{pmatrix}$$

Both systems are stable, share the same eigenvalues, but stability depends on switching!



Clarification: Switching between…





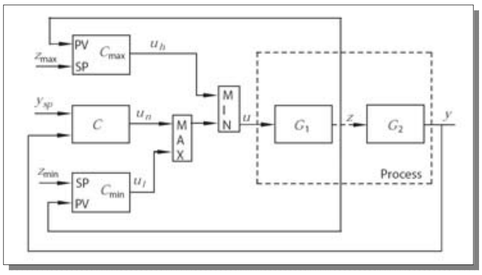
Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

Constrained control via min-max selectors

Common “pre-HYCON” approach for constrained control

Aim: tracking of primary variable (y), while keeping secondary variable (z) within limits



[Johansson, 2002]

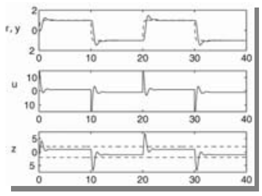
Numerical example

Specific example with

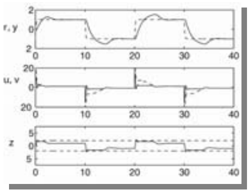
$$G_1(s) = \frac{40}{0.5s^3 + 2s^2 + 22s + 40} \qquad G_2(s) = \frac{5}{s^2 + 7s + 5} \qquad C(s) = \frac{s^2 + 3s + 3}{0.02s^2 + s + 0.01}$$

and proportional constraint controllers.

Control without constraint handling



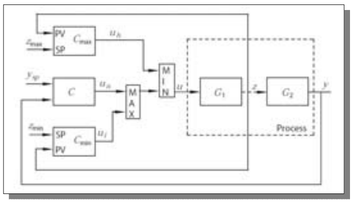
Control with constraint handling

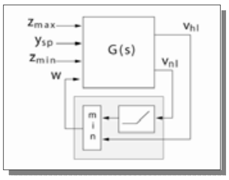


A loop transformation

Closed loop: linear system interconnected with 3-input/1-output static nonlinearity

Loop transformation reduces dimension of nonlinearity by one:

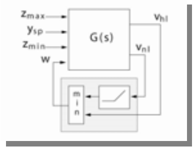


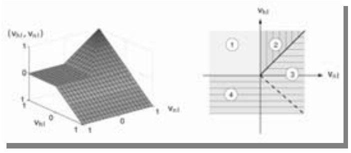


still, few techniques apply to such systems...(small gain, LDI, for example, don't work)

Stability analysis

However, nonlinearity (and hence system) is piecewise linear:





LMI computations return quadratic Lyapunov function (but S-procedure needed)

Part III – Examples

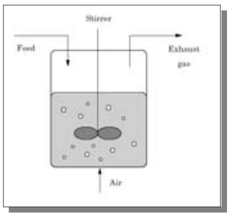
- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

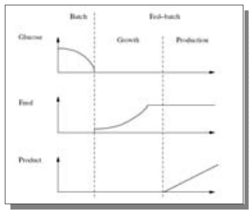
Fed-batch cultivation of *E. coli*

Recombinant (genetically modified) *E. coli* bacteria used to produce proteins.

Bioreactor operation: Feed (nutrition) and oxygen added to maximize cell growth.

Fed-batch: feed added continuously, at limiting rate

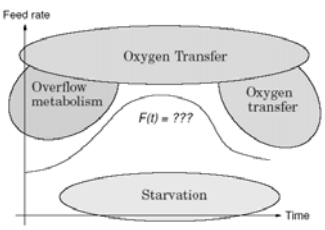




[Velut, 2005]

Control objective

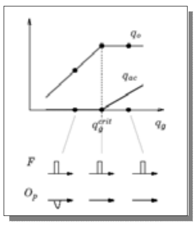
Objective: maximize feed rate while ensuring that
- oxygen level does not drop too low (acetate production, inhibited growth)
- glucose is not in excess ("overflow metabolism")

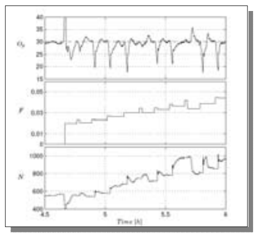


Probing control

Control strategy: increase feed while no acetate is formed, decrease otherwise

Acetate formation detected by probing: add pulse in feed, observe if oxygen consumed





A piecewise linear abstraction

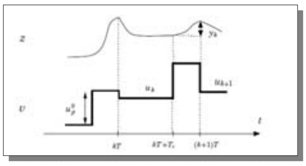
Simplified model of reactor dynamics

$$\dot{x} = ax + bf(v)$$
$$y = cx$$

where $f(v)$ is a piecewise linear function and

$$v(t) = u_k + u_p(t) \quad t \in [kT, (k+1)T]$$

and r is a static reference.



Integrating the response over a pulse period, we find the discrete-time model

$$x[k+1] = Ax[k] + B\begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix}$$
$$y[k] = Cx[k] + D\begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix}$$

Piecewise linear if u_k is a linear in x.

Control strategy

Assume a linear integral control

$$u[k+1] = u[k] + K(y_{ref}[k] - y[k])$$

fixed length of probing cycle T and probing pulse $T - T_c$

To model saturation in glucose uptake, consider

$$f(v) = \min(v, r^*)$$

This results in a piecewise linear systems with three regions (why not two?)

Control objective is now to drive system towards saturation.

Control to saturation

The formulation in Lecture 2 does not return any feasible solution
- reason: integrator dynamics in unbounded regions → not exponentially stable

Two potential approaches:

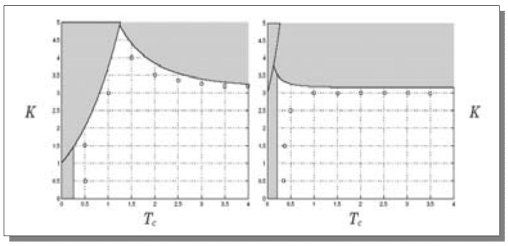
- Prove convergence for initial values within (hopefully large but bounded) region (can be done by adding S-procedure terms)

- Remove implicit equality constraints by state-transformation (more satisfying, but more complex; see Velut)

With modifications, stability can (often) be proven using pwq Lyapunov functions.

Numerical results

Stability regions for one specific problem instance (reactor parameters)
- red dots bound region where stability can be established numerically
- shaded regions are shown to be unstable (via local analysis)



Performance analysis

Stability typically not enough with stability – would like to optimize performance
- for example, the ability to track time-varying saturation level

Can compute bound γ on performance

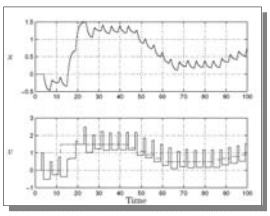
$$\sum_k (\bar{x}[k] - \bar{x}_{ref}[k])^T Q(\bar{x}[k] - \bar{x}_{ref}[k]) \le \gamma^2 \sum_k (r[k+1] - r[k])^2$$

for all reference trajectories r[k] via LMI computations.

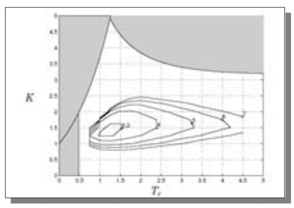
Note: typically large system descriptions...

Numerical example

Simulations for specific r[k]



γ for all rate-limited references



Parameter contours suggest optimal parameters $K \approx 1.4,\ T_c \approx 1.3$

Tuning rules

Similar behavior can be observed for various parameter values of the process.

Based on this observation, Velut suggests the following tuning rules

$$K = \frac{1}{\sigma(T - T_c)}$$
$$y_{ref} = \frac{\sigma(T - T_c)}{2} u_p^0$$
$$1 < \alpha T_c < 2$$

where $\sigma(t)$ is the unit step response of the linear dynamics.

Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

A simple model for car dynamics

Simple model:

$$M\dot{v} = F - F_l \quad \text{car dynamics}$$
$$F_l = kv^2 \text{sign}(v) - Mg\sin\alpha \quad \text{load force}$$
$$F = p/rT \quad \text{gear box relations}$$
$$v = r/p\omega \quad \text{gear box relations}$$

Inputs: motor torque T and road incline α; output ω

$$\dot{v} = \frac{1}{M}Tu - \frac{k}{M}v^2 \text{sign}\, v - g\sin\alpha$$
$$\omega = vu$$

where $u = p/r$ is the discrete input, determined by the current gear

To emphasize this dependence, we write

$$u = u_i := p_i/r_i \quad \text{when using gear } i$$

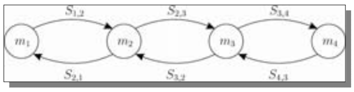
[Pettersson, 1999]

Gear-switching

Gear-switching strategy:

$$i(t^+) = i(t) + 1 \quad \text{if } \omega > \omega_{i(t)}^{hi}$$
$$i(t^+) = i(t) - 1 \quad \text{if } \omega < \omega_{i(t)}^{lo}$$

Can be represented by hybrid automaton with four discrete states



Torque control and bumpless transfer

Base controller: non-linear PI

$$T = P + I + \frac{k}{u_i}v^2 \text{sign}\, v$$
$$P = K_{i(t)}(v_{ref} - v)$$
$$\frac{d}{dt}I = \frac{K_{i(t)}}{T_{i(t)}}(v_{ref} - v)$$

Abrupt changes in acceleration when changing gears avoided via bumpless transfer:

$$u_i K_i = u_j K_j$$
$$I(t^+) = \frac{u_{i(t)}}{u_{i(t^+)}} I(t)$$

for all feasible gear changes i→j. (compatible values of K_i, changes in integral state)

Hybrid system model

Need extended hybrid model that allows for state jumps in the continuous state

$$\dot{x}(t) = f(x(t), i(t))$$
$$x(t^+) = \rho(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

LMI formulation possible if jump map is affine in x.

Numerical example

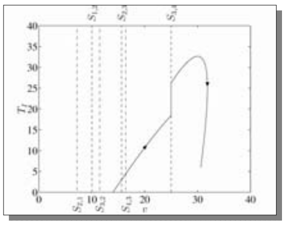
Closed loop system is switched linear system

$$\frac{d}{dt}\begin{bmatrix} e \\ I \end{bmatrix} = \begin{bmatrix} u_i K_i/M & p_i/M \\ K_i/T_i & 0 \end{bmatrix}\begin{bmatrix} e \\ I \end{bmatrix} \quad \text{for } e \in X_i$$

where $e = v_{ref} - v$ and

$$u_i = \{50, 32, 20, 14\}$$
$$K_i = \{3.75, 5.86, 9.37, 13.39\}$$
$$M = 1500,\ T_i = 40,\ T_i K_i = 187.5$$

Simulation for $v_{ref} = 30$



Stability

If affine reset maps

$$x(t^+) = H_{i(t)i(t^+)}\bar{x}(t)$$

then, condition $\bar{x}(t^+)^T \bar{P}_j \bar{x}(t^+) < \bar{x}(t)^T \bar{P}_i \bar{x}(t)$ is guaranteed by solution to LMI

$$0 < \bar{P}_i - H_{jk}^T \bar{P}_k H_{jk} + f_{jk}^T t_{jk} + t_{jk}^T f_{jk}$$

Allows extension of discontinuous Lyapunov function computations from Lecture 2.

Gear-box example: solution found → exponential convergence to v_{ref}

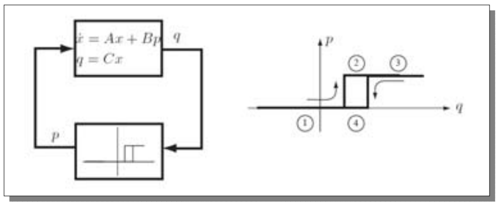
Remark: analysis needs to be repeated for each value of v_{ref} (compare bioreactor example)

Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

More of a theoretical challenge…

Consider a linear control system under hysteresis relay feedback…



[Hassibi, 2000]

$$A = \begin{pmatrix} -0.1 & -1 \\ 0 & -0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Extensive simulations suggest system is stable, yet no pwq Lyapunov function found.

The challenge

Question: why do piecewise quadratic methods fail, and how can they be improved?

The more general challenge:

Put the methods to the test of challenging engineering problems, and help to contribute to the development to improved analysis tools!

References

M. Johansson, "Piecewise linear control systems – a computational approach", Springer Lecture Notes in Control and Information Sciences no 284, 2002.

S. Velut, "Probing control – analysis and design with application to fed-batch bioreactors", PhD thesis, Lund University, Lund, Sweden, June 2005.

S. Pettersson, "Analysis and design of hybrid systems", PhD thesis, Chalmers University, Gothenburg, Sweden, 1999.

A. Hassibi, "Lyapunov methods in the analysis of complex dynamical systems", Stanford University, Stanford, CA, 2000.