Progress on the reachability analysis and verification methods for hybrid systems

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Reachability analysis

Reachable set computations are useful for

- **Verification**
  problems such as proving that the system does not reach a 'bad' state
- **Controller synthesis**
  problems such as determining the set of states from which it is possible to reach a target set while avoiding a forbidden set

Many existing methods and tools (see the next slide)

Reachability analysis

Direct methods

- Track the evolution of the reachable set under the flow of the system. Various set representations: *e.g.* polyhedra, ellipsoids, level sets
- Exact results, or accurate approximations with error bounds. Using symbolic or numerical computations
- Tools: Coho, CheckMate, d/dt, HysDel, VeriShift, Vertdict, Requiem, Level-set toolbox, ..

Indirect methods

- Abstraction methods: reducing to a simpler system that preserves the property (*e.g.* Tiwari & Khanna 02; Alur et al. 02; Clarke et al. 03)
- Prove the property without computing reachable sets: *e.g.* Barrier certificates Prajna & Jadbabaie04, polynomial invariants Tiwari & Khanna04.

⋆ **Scalability** is still challenging (complexity and size of real-life systems)

Our progress in reachability analysis

Accurate approximations

- Complexity of the dynamics
  - Hybridization methods for non-linear systems
  - Extension to differential algebraic systems
- Size of the system
  - Reachability technique using zonotopes $\Rightarrow$ large scale systems

Abstraction methods: predicate abstraction, projection

Plan

- Hybridization methods for non-linear systems
- Extension to differential algebraic systems
- Reachability computations using zonotopes
- Abstraction by projection

Plan

- **Hybridization methods for non-linear systems** [Asarin, Dang, Girard 03, 05]
- Extension to differential algebraic systems
- Reachability computations using zonotopes
- Abstraction by projection

Hybridization: Principle

System Δ : $\dot{x} = f(x)$, $x \in \mathcal{X}$, f is Lipschitz

Step 1: Construction of the approximate system:

- Partition the state space $\mathcal{X}$ into disjoint regions of size $\mathbf{h}$ and assign to each region an approximate vector field
- $\mathbf{h}$: space discretization size
- $f_{\mathbf{h}}$: resulting vector field over the whole state space $\mathcal{X}$
- Approximation error $\boldsymbol{\varepsilon}(\mathbf{h}) = \sup_{x \in \mathcal{X}} ||\mathbf{f}(\mathbf{x}) - \mathbf{f}_{\mathbf{h}}(\mathbf{x})||$
- Conservative approximate system

$$\text{System } \Delta_{\mathbf{h}} : \quad \dot{x} = f_{\mathbf{h}}(x) + u$$

$u(\cdot)$: disturbance taking values in $Ball(\boldsymbol{\varepsilon}(\mathbf{h}))$

Hybridization: Principle (cont'd)

Step 2. Using $\Delta_{\mathbf{h}}$ to yield approximate analysis results for Δ

Convergence results: If $\Delta_{\mathbf{h}}$ is continuous

- The distance between the reachable sets $d_H\left(Reach(\Delta), Reach(\Delta_{\mathbf{h}})\right)$ is $\mathcal{O}(\varepsilon(\mathbf{h}))$
- The reachable set of $\Delta_{\mathbf{h}}$ converges to the reachable sets of Δ with the same rate as $f_{\mathbf{h}}$ converges to f

We developed two methods for constructing approximate systems with good error bound $\varepsilon(\mathbf{h})$

- Piecewise affine systems
- Piecewise multi-affine systems

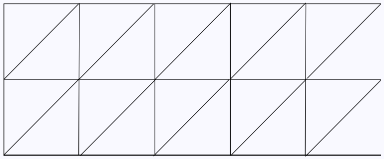
Piecewise affine approximation

- Using a simplicial mesh, each cell C_i is a simplex of size $\mathbf{h}$ (edge length)
- Define for each C_i a linear function $f_{\mathbf{h}}$ interpolating f at its vertices
- Piecewise linear function $f_{\mathbf{h}}$ is continuous over the state space

Approximation error

If f is C^2 on $\mathcal{X}$ with bounded second order derivatives $\Rightarrow$ **quadratic error**: $\varepsilon(\mathbf{h}) = \mathcal{O}(\mathbf{h}^2)$.

Mesh construction: decompose a hypercube into $n!$ simplices

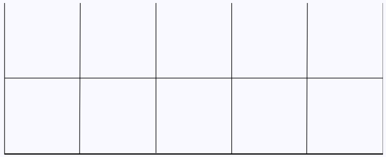


- Reachability computations for $\Delta_{\mathbf{h}}$: various existing techniques
- Our implementation using reachability procedures of the tool **d/dt**

Piecewise multi-affine approximation

- Using a rectangular mesh, each cell C_i is a hypercube of size $\mathbf{h}$
- Define for each cell C_i a multi-linear function $f_{\mathbf{h}}$ interpolating f at its vertices $\Rightarrow$ iteratively applying linear interpolation on each dimension
- Piecewise multi-linear function $f_{\mathbf{h}}$ is continuous over the state space

Approximation error: If f is C^2 on $\mathcal{X}$ with bounded second order derivatives $\Rightarrow$ **quadratic error**: $\varepsilon(\mathbf{h}) = \mathcal{O}(\mathbf{h}^2)$.



Piecewise multi-affine approximation (cont'd)

⋆ Advantage comparison

Simplicial meshes	**Rectangular meshes**
	smaller number of cells
	less complex geometric structure
available techniques for approximate systems	???

⋆ Reachability computations for piecewise multi-affine systems with input

- Use projection to obtain a uncertain bilinear control system
- Then, use our reachability technique for bilinear control systems

Plan

- Hybridization methods for nonlinear systems
- **Extension to differential algebraic systems** [Dang, Donze, Maler FMCAD04]
- Reachability computations using zonotopes
- Abstraction by projection

Differential Algebraic Equations

Motivations

- DAEs arise in numerous applications: *e.g.* electrical circuits, constrained mechanical systems, chemical reaction kinetics, singular perturbation problems
- Our interest in applications of hybrid systems techniques to verification of analog and mixed-signal circuits

Reachability analysis of DAEs

$$F(x, \dot{x}) = 0$$

- DAEs differ from ODEs (in theoretical and numerical properties)
- Differential index: minimal number of differentiations required to solve for the derivatives $\dot{\mathbf{x}}$
- We focus on **DAEs of index 1**

Reachability analysis of DAEs (cont'd)

We study the equivalent semi-explicit form:

$$\begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y) \end{aligned}$$

- Transforming into ODEs :
  Differentiating the algebraic eq. once gives $\dot{y} = -g_y^{-1} g_x f$ where $g_y(x,y) = \partial g/\partial y$. (Note that the DAEs are of index 1)
  $\Rightarrow$ Obtain augmented ODEs with variables $z = (x,y)^T$:

$$\dot{z} = (f, -g_y^{-1} g_x f)^T = \tilde{f}$$

- Retain the algebraic constraint and intepret the original DAEs as the augmented ODEs on a manifold :

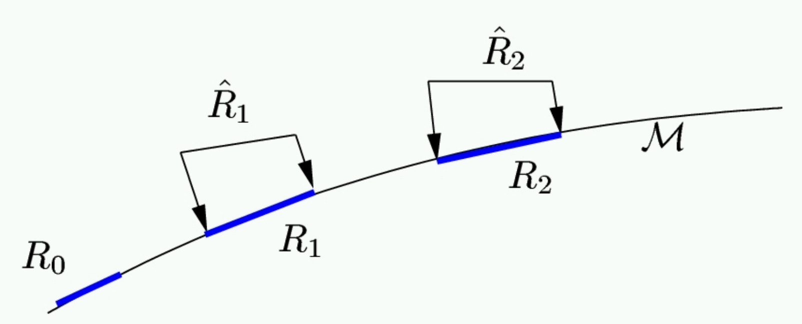
$$\begin{aligned} \dot{z} &= \tilde{f}(z) \\ 0 &= g(z) \end{aligned}$$

ODEs on manifolds

Remark: ODEs on manifolds are useful to study systems with invariants

$$\begin{aligned} \dot{z}(t) &= f(z(t)) \\ 0 &= g(z(t)) \quad \Rightarrow \textit{defining a manifold } \mathcal{M} \\ z(0) &\in R_0 \end{aligned}$$

Combining reachability computations techniques for ODEs and ideas from *geometric integration using projection* [Lubich,Hairer,Wanner 2003]



Algorithm for ODEs on manifolds

R_0: initial set
repeat $k = 0, 1, \ldots$
$\hat{R}_{k+1} = \mathbf{Reach}_{[0,r]}(R_k)$ /* *computed for the augmented ODEs* */
$R_{k+1} = \mathbf{\Pi}_{\mathcal{M}}(\hat{R}_{k+1})$ /* *project on the manifold* $\mathcal{M}$ */
until $R_{k+1} = \bigcup_{i=1}^{k} R_i$

- Projection:

$$\mathbf{\Pi}_{\mathcal{M}}(\hat{z}) = \arg\min_{z} |\hat{z} - z| \text{ subject to } g(z) = 0$$

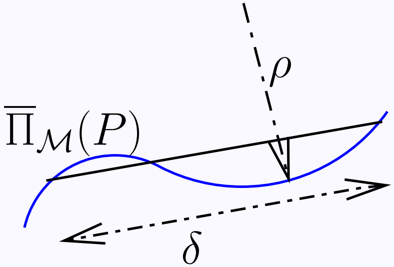
- Convergence : same order as the convergence order of the technique for ODEs (projection does not deteriorate the convergence)
- Second order method

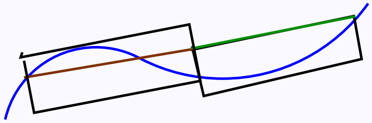
Approximation of the projection

$$\text{Manifold } \mathcal{M}: \quad g(x) = 0$$

$$P \text{ is a convex polyhedron, } \quad \text{computing } \mathbf{\Pi}_{\mathcal{M}}(P)??$$

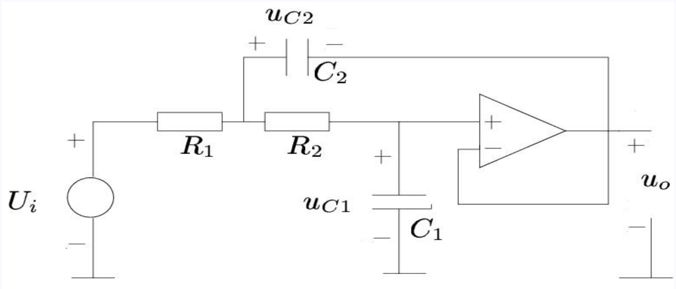
- If the algeb. constraint is linear, $\mathbf{\Pi}_{\mathcal{M}}$ is computed using linear algebra.
- $\{v^1, \ldots, v^m\}$: vertices of P, $\overline{\Pi}_{\mathcal{M}}(P) = conv\{\mathbf{\Pi}_{\mathcal{M}}(v^1), \ldots, \mathbf{\Pi}_{\mathcal{M}}(v^m)\}$.
- Using $\overline{\Pi}_{\mathcal{M}}(P)$ to over-approximate the projection
  – Estimate ρ, the maximum radius of curvature of $\mathcal{M}$ for $x \in \overline{\Pi}_{\mathcal{M}}(P)$
  – Estimate the diameter δ of $\overline{\Pi}_{\mathcal{M}}$
  – If $\rho \leq \kappa\delta$, subdivide $\overline{\Pi}_{\mathcal{M}}(P)$ and then repeat the procedure for each subpolyhedron. Otherwise, find a polyhedron enclosing $\mathbf{\Pi}_{\mathcal{M}}(P)$.





Example: Biquad lowpass filter

[Hartong,Hedrich,Barke 2002]



$$\dot{u}_{C1} = \frac{u_{C2} + u_o - u_{C1}}{C_1 R_2} \qquad \dot{u}_{C2} = \frac{U_i - u_{C2} - u_o}{C_2 R_1} - \frac{u_{C2} + u_o - u_{C1}}{C_2 R_2}, \tag{1}$$

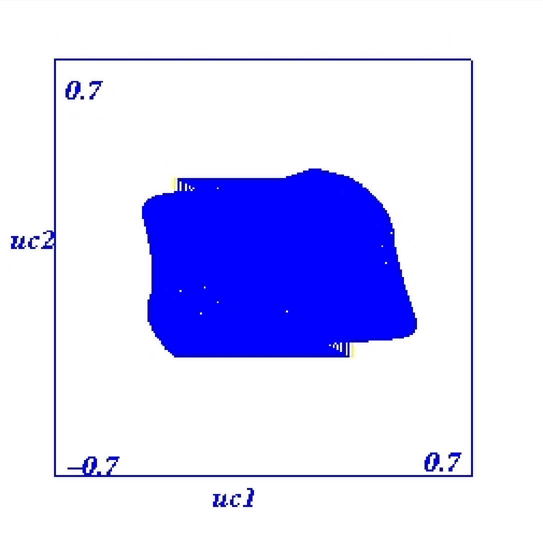
$$u_o - V_{max} \tanh\left(\frac{(u_{C2} - u_o)V_e}{V_{max}}\right) + U_{om} = 0, \tag{2}$$

$$U_{om} = \mathcal{V}(i_0), \quad i_o = -C_2\, \dot{u}_{C2}, \tag{3}$$

$$\mathcal{V}(i_o) = K_1 i_o + 0.5\sqrt{K_1 i_o^2 - 2K_2 i_o I_s + K_1 I_s^2 + K_2} - 0.5\sqrt{K_1 i_o^2 + 2K_2 i_o I_s + K_1 I_s^2 + K_2}. \tag{4}$$

Biquad lowpass filter: verification results

The property to verify is the ***absence of overshoots*** .



- $C_1 = 0.5e-8$, $C_2 = 2e-8$, and $R_1 = R_2 = 1e6$ (highly damped case)
- The initial set: $u_{C1} \in [-0.3, 0.3]$, $u_{C2} \in [-0.3, 0.3]$ and $u_o \in [-0.2, 0.2]$
- Reachability for the ODE part is done using a simplicial mesh

Plan

- Hybridization methods for nonlinear systems
- Extension to differential algebraic systems
- **Reachability computations using zonotopes** [A. Girard 2005]
- Abstraction by projection

Linear Systems with uncertain inputs

$$\dot{x} = Ax + u, \quad ||u(\cdot)|| \leq \mu$$

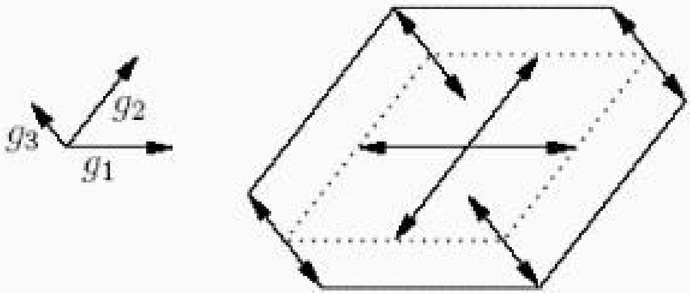
- $Reach_r(X_0) \subseteq \mathbf{e^{rA}} X_0 + Ball(\alpha_r)$
- $\alpha_r = \dfrac{e^{r||A||} - 1}{||A||}\mu$
- Two required operations:
  - **Linear operator** $\mathbf{e^{rA}}$
  - **Minkowski sum** ('expanding' the reachable set of the autonomous system by α_r)
- On **zonotopes**, these two operations can be efficiently performed (see next)

Zonotopes

- Zonotope: Minkowski sum of a finite number of segments:

$$Z = \{x \in \mathbb{R}^n \mid x = \mathbf{c} + \sum_{i=1}^{p} x_i \mathbf{g_i}, \quad -1 \le x_i \le 1\}.$$

- $\mathbf{c}$ is the center of the zonotope, $\{\mathbf{g_1}, \dots, \mathbf{g_p}\}$ are the generators. The ratio p/n is the order of the zonotope.



Two-dimensional zonotope with 3 generators

Computational advantages of zonotopes

- Encoding of a zonotope has a **polynomial complexity** wrt dimension (vs. **exponential complexity** for general convex polyhedra)
- Zonotopes are closed under **linear transformation**

$$Z = (\mathbf{c}, \langle \mathbf{g_1}, \dots, \mathbf{g_p} \rangle)$$

$$LZ = (L\mathbf{c}, \langle L\mathbf{g_1}, \dots, L\mathbf{g_p} \rangle)$$

- Zonotopes are closed under the **Minkowski sum**

$$\mathbf{Z_1} = (\mathbf{c_1}, \langle \mathbf{g_1}, \dots, \mathbf{g_p} \rangle), \quad Z_2 = (c_2, \langle h_1, \dots, h_q \rangle)$$

$$\mathbf{Z_1} + Z_2 = (\mathbf{c_1} + c_2, \langle \mathbf{g_1}, \dots, \mathbf{g_p}, h_1, \dots, h_q \rangle)$$

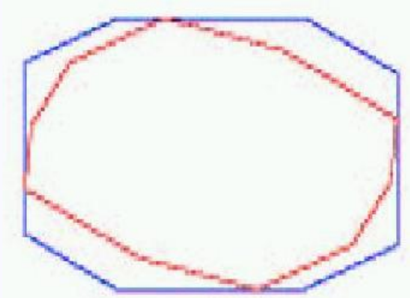
$\Rightarrow$ **Important properties** needed for reachability computations

Complexity reduction

At each iteration, the order of the zonotope increases (due to the Minkowski sum) $\Rightarrow$ Complexity is $\mathcal{O}(\mathbf{N^2})$ where $\mathbf{N}$ is the number of iterations

Controlling the order growth

- When the order is greater than m, over-approximate by a zonotope of lower order $\Rightarrow$ Efficient zonotope order reduction techniques exist
- Thus, the complexity of the algorithm is $\mathcal{O}(N)$

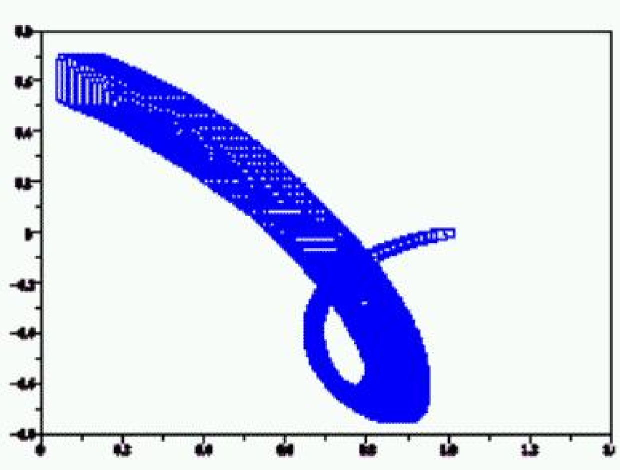


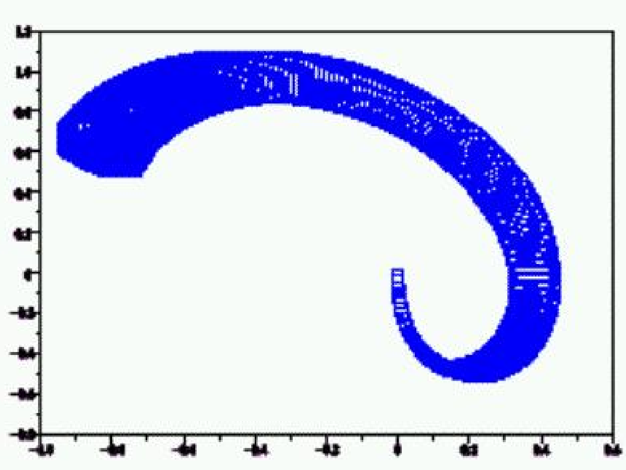
Performance

Dimension	5	10	20	50	100
CPU time (s)	0.05	0.33	1.5	9.91	43.7

(Computation of $Reach_{[0,1]}$, 100 iterations, zonotope order=5)

A 5-dimensional system





Projections of $Reach_{[0,1]}$, 200 iterations, order of the zonotopes 40.

Reachability computations using zonotopes: Summary

- **Efficient** and **scalable**
- Handle systems up to **100 dimensions**
- Can be extended to **non-linear** systems and **hybrid** systems
- **Future work:** Computational methods for zonotopes (intersection, union)

Plan

- Hybridization methods for nonlinear systems
- Extension to differential algebraic systems
- Reachability computations using zonotopes
- **Abstraction by projection** [Asarin & Dang 04]

Introduction

- Basic idea: **project away** some variables the evolution of which is modeled as input
- **Dimension reduction** method for continuous systems
- A 'hybridization' method using ideas of qualitative simulation
- **Goals:**
  - **more precise than qualitative simulation**
  - **less expensive than analyzing the original system** (due to lower dimension)

Principle

$$\begin{cases} \dot{x} & = f(x, y, z) \\ \dot{y} & = g(x, y, z) \\ \dot{z} & = h(x, y, z) \end{cases}$$

- We want to abstract away variable **z**
- Partition the domain of **z** into k disjoint intervals

$$\{[l^1, u^1), [l^2, u^2), \ldots [l^k, u^k]\}$$

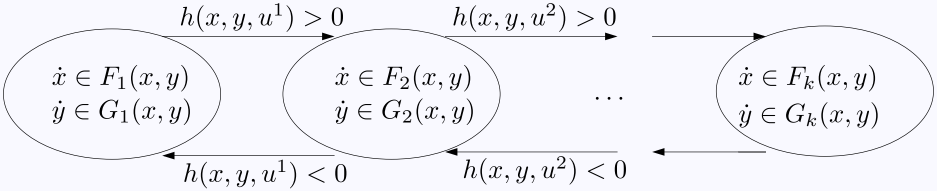
where $l^{i+1} = u^i$ for all i

- If $z \in I_z^i = [l^i, u^i]$, the dynamics of x and y can be approximated by **differential inclusion** :

$$\begin{cases} \dot{x} & \in F_i(x, y) = \{f(x, y, z) \mid z \in I_z^i\} \\ \dot{y} & \in G_i(x, y) = \{g(x, y, z) \mid z \in I_z^i\} \end{cases}$$

Hybridization

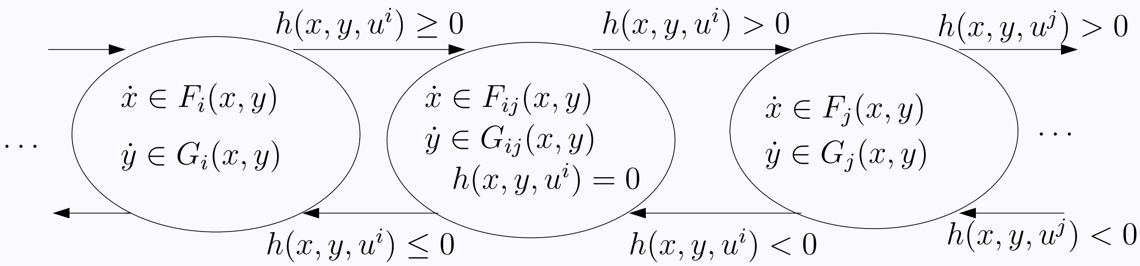
- The original system is thus approximated by **2**-dimensional **hybrid** system with **k** different continuous dynamics
- **Switching** between **adjacent intervals** I_z^i:
  - Transition from $I_z^i = [l^i, u^i)$ **to** $I_z^{i+1} = [l^{i+1}, u^{i+1})$ is possible if at the boundary the derivative of **z** is positive, i.e. $h(x, y, u_i) > 0$
  - Similarly, transition from I_z^{i+1} **to** I_z^i if $h(x, y, u_i) < 0$
  - These switching conditions **are not sufficient** $\Rightarrow$ **conservative approximation**



Remedy Discontinuities

- Our hybridization method introduces **discontinuities**
- "**Convexify**" the dynamics at switching surfaces (to guarantee existence of solution, error bound)
- Between adjacent intervals I_z^i and I_z^j $(j = i + 1)$, add a location:

$$\begin{cases} \dot{x} & \in F_{ij}(x,y) = co\{F_i(x,y), F_j(x,y)\} \\ \dot{y} & \in G_{ij}(x,y) = co\{G_i(x,y), G_j(x,y)\} \end{cases}$$



Convergence result

- Resulting abstract system is upper semi-continuous and one-sided Lipschitz

  $\Rightarrow$ We can prove **error bound**:

  - Distance between trajectories of the original system and the abstract system is $\mathcal{O}(\delta)$
  - δ: bound on the distance between the derivatives (which depends on the size of the $\mathbf{z}$-mesh)

- **First order** method

Abstraction with timing information

- So far, only the sign of $\dot{z}$ is used to determine switching conditions
- The time the system can stay with a dynamics is omitted
- Inlude timing information to obtain more precise abstraction
  - Additionally **discretize derivatives** $\dot{z}$ into disjoint intervals
  - Each location corresponds to an interval I_z^i **of** z and an interval $I_{\dot{z}}^j$ **of** $\dot{z}$
  - Then, we can estimate **bounds on the staying times** $\Rightarrow$ embed in the switching conditions.

Computation Issues

- Linear Systems: abstract system is a linear system with uncertain input.
- Non-linear systems: abstract system is a **general differential inclusions**
- We focus on the case of **multi-affine systems** (which have numerous applications in biology, economy)

Abstraction of multi-affine systems

Given a system

$$\begin{cases} \dot{x}_1 & = a_1 x_1 + b_1 x_2 + c_1 x_1 x_2 \\ \dot{x}_2 & = a_2 x_1 + b_2 x_2 + c_2 x_1 x_2 \end{cases}$$

Abstract away $x_2 \Rightarrow$ Dynamics of each cell:

$$\begin{cases} \dot{x}_1 & = a_1 x_1 + b_1 \mathbf{u} + c_1 \mathbf{u} x_2 \\ ||\mathbf{u}(\cdot)|| \leq \mu \end{cases}$$

$\Rightarrow$ bilinear control system

Reachability analysis of Bilinear Control Systems

A bilinear control system with additive and multiplicative inputs

$$\dot{x}(t) = f(x(t), u(t)) = Ax(t) + \sum_{j=1}^{l} u_j(t) B_j x(t) + Cu(t)$$

Basic idea: Applying the Maximum principle to find 'optimal' input $\tilde{u} \Rightarrow$ require solving an optimal control problem for a bilinear system.

For tractability purposes,

1. Restrict to piecesiwe constant inputs
2. To solve bilinear diff equations, treat the bilinear term as independent input (see next)

Applying the Maximum Principe

⋆ Represent the initial set X_0 as **intersection of half-spaces**.

⋆ For each half-space $H = (q, x)$ with **normal vector** q and **supporting point** x.

$$\begin{array}{rcl} \dot{\tilde{x}} & = & A\tilde{x} + \tilde{u}B\tilde{x} + C\tilde{u} \\ \dot{\tilde{q}} & = & -\dfrac{\partial H}{\partial x}(\tilde{x}, \tilde{q}, \tilde{u}) \ \text{ where } H(q, x, u) = \langle q, Ax + ubx + cu \rangle \\ \tilde{u}(t) & \in & argmax\{\langle \tilde{q}(t),\ uB\tilde{x}(t) + Cu \rangle \mid u \in U\} \end{array}$$

with initial conditions: $\tilde{q}(0) = q, \ \ \tilde{x}(0) = x$.

Then,

- For all $t > 0$, the half-space $H(\tilde{q}(t), \tilde{x}(t))$ **contains** $Reach_t(X_0)$
- Its hyperplane is a supporting hyperplane of $Reach_t(X_0)$.

Bilinear Control Systems

⋆ Solving the optimal control problem for arbitrary inputs is hard ⇒ restrict to **piecewise constant inputs** $u(t) = u^k$ for $t \in [t_k, t_{k+1})$.

⋆ Solving bilinear systems with piecewise constant input: r is **time step**

$$x^{k+1} = e^{Ah} x^k + \int_0^r e^{A(r-\tau)} u^k b\, \mathbf{x}(\tau)\, d\tau + \int_0^r e^{A(r-\tau)} c u^k\, d\tau$$

- Approximate $x(\tau)$ for $\tau \in [0, r)$ by: $\pi(\tau) = \alpha\tau^3 + \beta\tau^2 + \gamma\tau + \sigma$ satisfying Hermite interpolation conditions: $\pi(0) = x(t_k)$, $\dot{\pi}(0) = \dot{x}(t_k)$, $\pi(r) = x(t_{k+1})$, $\dot{\pi}(r) = \dot{x}(t_{k+1})$
- Replacing $\mathbf{x}(\tau)$ by $\pi(\tau)$ in the integral, we obtain: $Mx^{k+1} = Dx^k + d$
- We can prove that the **error is quadratic** in time step $O(r^2)$

Example: A biological system

A multi-affine system, used to model the gene transcription control in the *Vibrio fischeri* bacteria [Belta et al 03].

$$\begin{cases} \dot{x}_1 & = & k_2 x_2 - k_1 x_1 x_3 + u_1 \\ \dot{x}_2 & = & k_1 x_1 x_3 - k_2 x_2 \\ \dot{x}_3 & = & k_2 x_2 - k_1 x_1 x_3 - n x_3 + n u_2 \end{cases} \tag{5}$$

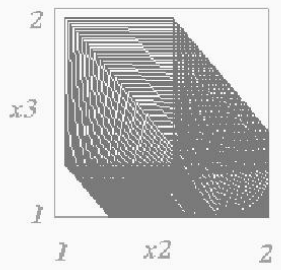
State variables x_1, x_2, x_3 represent cellular concentration of different species
Parameters k_1, k_2, n are binding, dissociation and diffusion constants.
Control variables u_1 and u_2 are plasmid and external source of autoinducer.

Goal: drive the system through to the face $x_2 = 2$

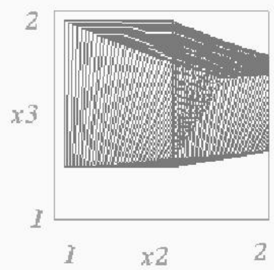
Example: A biological system (cont'd)

Results obtained by abstracting away x_1. Location $x_1 \in [1.0, 1.5]$

uncontrolled system $(u = 0)$



controlled system



Ongoing and Future work

- Zonotopic calculus
- Efficient method for multi-affine systems
- Hybridization: Hierarchical mesh construction
- Randomized simulation with coverage criteria
- Guided abstraction refinement