Robust MPC and multiparametric convex programming

A. Bemporad
C. Filippi
D. Muñoz de la Peña

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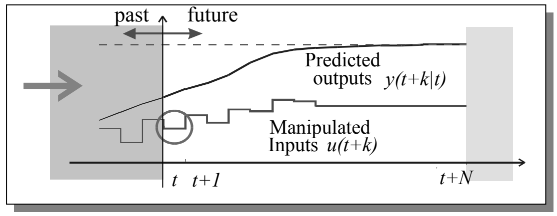
Outline

- Motivation: Robust MPC
- Multiparametric convex programming
- Kothares'96
- Examples

Model Predictive Control



<u>At time t:</u>

- Get new measurements $y(t)$
- Estimate the current state $x(t)$
- Solve with respect to $[u'(t), \ldots, u'(t+N-1)]'$ the QP prob. (=finite-horizon open-loop optimal control)
- Apply only $u(t) = u^*(t)$ and discard the remaining optimal inputs
- Go to time $t+1$

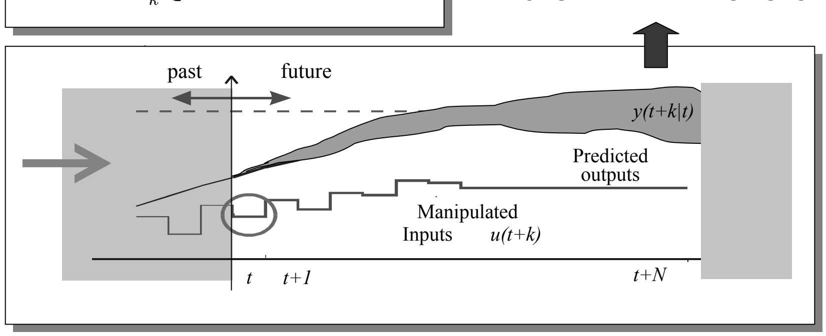
Robust MPC

Robust MPC are based on uncertain models:

$$x_{k+1} = A(w_k)x_k + B(w_k)u_k + v_k$$

$$w_k \in W$$
$$v_k \in V$$

UNCERTAIN PREDICTIONS



Robust MPC

Robust MPC formulation:

$$J^*(x) = \min_{\mathbf{u}} \max_{\mathbf{w} \in W_N} V(x, \mathbf{u}, \mathbf{w})$$

subject to:

$$\begin{aligned} x_j = x_j(x, \mathbf{u}, \mathbf{w}) &\in X, & \forall \mathbf{w} \in W_N, \quad \forall j \\ x_N = x_N(x, \mathbf{u}, \mathbf{w}) &\in \Omega, & \forall \mathbf{w} \in W_N \\ u_j = u_j(x, \mathbf{u}, \mathbf{w}) &\in U, & \forall \mathbf{w} \in W_N, \quad \forall j \end{aligned}$$

Worst case minimization

Robust constraint satisfaction

Robust MPC

Robust MPC computation:

$$J^*(x) = \min_{\mathbf{u}} \max_{\mathbf{w} \in W_N} V(x, \mathbf{u}, \mathbf{w})$$

In general, $V(x, \mathbf{u}, \mathbf{w})$ is convex on $x, \mathbf{u}, \mathbf{w}$.

$$\widehat{V}(x, \mathbf{u}) = \max_{\mathbf{w} \in W_N} V(x, \mathbf{u}, \mathbf{w})$$

Maximum convex function

HARD

$$J^*(x) = \min_{\mathbf{u}} \widehat{V}(x, \mathbf{u})$$

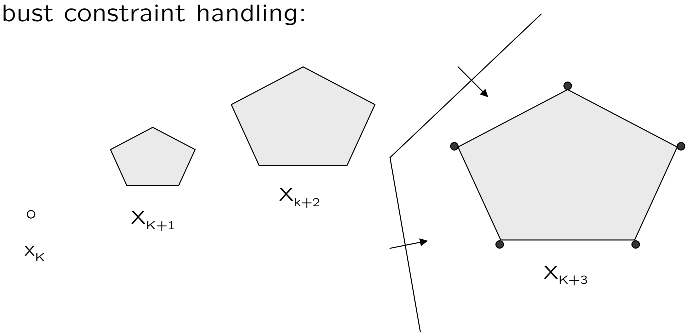
Minimum convex function

Tractable

HIGH COMPUTATINAL BURDEN

Robust MPC

Robust constraint handling:



One constraint for each vertex on the convex hull

Exponential number of constraints with the prediction horizon

Robust MPC

Open loop MPC:
Minimize over a single trajectory

Number of free variables linear with the prediction horizon

Feedback:
Minimize a set of trajectories

Number of free variables exponential with the prediction horizon
Apply dynamic programming

Conclusion

MPC ROBUST
(in general)
ARE
COMPUTATIONALY
DEMANDING

Robust MPC & mp Convex

Robust MPC controllers often are defined as convex optimization problems

$$\begin{array}{ll} \min_x & f(x,\theta) \\ \text{s.t.} & g_i(x,\theta) \leq 0 \quad (i=1,\ldots,p) \\ & Ax + B\theta + d = 0 \end{array}$$

Tractable but not solvable on line

Linear Cost function
Explicit solution using Dynamic programming

[2] A. Bemporad, F. Borreli, M.Morari "Min-max control of constrained uncertain discrete time linear systems",IEEE Trans. Automatic Control 2003.

Quadratic cost function
Not extended

Multiparametric
Convex
Programming

Multiparametric Convex Programming

Computation offline

Suboptimal solution

Online efficienty

IMPLEMENTATION

Multiparametric Convex Programming

$$\begin{aligned} \min_x \quad & f(x,\theta) \\ \text{s.t.} \quad & g_i(x,\theta) \le 0 \quad (i = 1,\dots,p) \\ & Ax + B\theta + d = 0 \end{aligned}$$

where:

- $x \in \mathbb{R}^n$ are the decision variables
- $\theta \in \Theta \subseteq \mathbb{R}^m$ are the parameters
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the objective function
- $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ define the inequality constraints

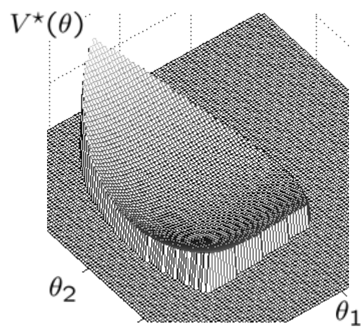
Assumption *Functions f and g_i $(i = 1,\dots,p)$ are jointly convex in both variables and parameters.*

Basic Result on Convex MP

$$\begin{aligned} \min_x \quad & f(x,\theta) \\ \text{s.t.} \quad & g_i(x,\theta) \leq 0 \quad (i = 1,\ldots,p) \\ & Ax + B\theta + d = 0 \end{aligned}$$

Lemma *Let f, g_i be jointly convex functions of (x,θ) $(\forall i = 1,\ldots,p)$. Then Θ_f is a convex set and $V^\star$ is a convex function of θ.*

(Mangasarian, Rosen, 1964)



V^* and Θ_f may not be easy to express analytically

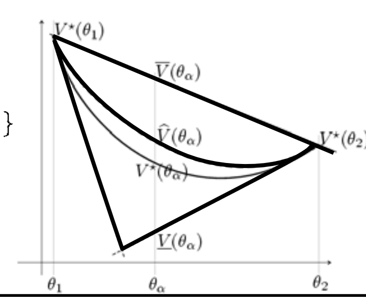
Bounds on the Value Function

- Consider the m-dimensional simplex: $S \triangleq \{\theta \in \mathbb{R}^m : \theta = \sum_{i=0}^{m} \mu_i \theta^i, \ \sum_{i=0}^{p} \mu_i = 1,\ \mu_i \geq 0\}$
- Optimizers at the vertices: $x_i \triangleq x^*(\theta_i)$
- Linear upper-bound: $\overline{V}(\theta) \triangleq \sum_{i=0}^{m} \mu_i f(x_i, \theta_i)$
- Define $\hat{x}(\theta) \triangleq \sum_{i=0}^{m} \mu_i x_i$. $\hat{x}(\theta)$ is feasible for all $\theta \in \Theta_f$
- Convex upper-bound: $\hat{V}(\theta) \triangleq f(\hat{x}(\theta), \theta)$
- Convex PWA lower-bound: (s^i=subgrad. @θ_i)

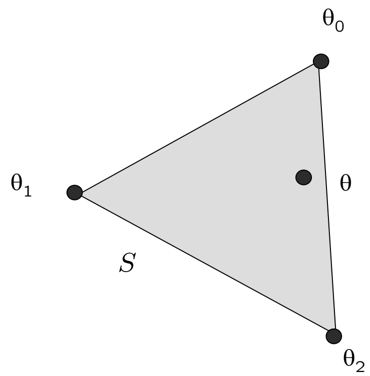
$$\underline{V}(\theta) \triangleq \max_{i=0,\ldots,m} \{V^\star(\theta^i) + (s^i)'(\theta - \theta^i)\}$$

- **Result**:

$$\begin{aligned} & \overline{V}(\theta) \geq \hat{V}(\theta) \geq V^\star(\theta) \\ & \underline{V}(\theta) \leq V^\star(\theta) \quad \text{for all } \theta \in S \end{aligned}$$



Linear feedback solution

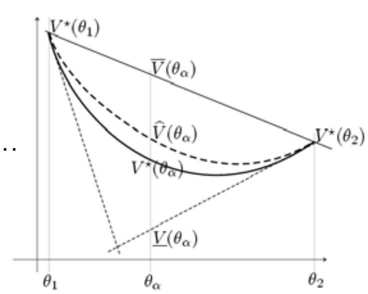


$$\widehat{x}(\theta) \triangleq \sum_{i=0}^{m} \mu_i x_i$$

$$\theta \in S,\ \widehat{x} = K_S\theta + q_S$$

LINEAR SOLUTION

Max Error-Bounds Computation



- Max absolute error inside simplex S:

  $$\epsilon^{MAX}(S) \triangleq \max_{\theta\in S}\{\widehat{V}(\theta) - V^\star(\theta)\}$$

  not very easy to compute in general ...

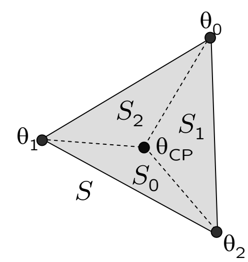
- Define another error measure:

$$\epsilon^{CP}(S) \triangleq \max_{\theta\in S}\{\overline{V}(\theta) - V^\star(\theta)\} = \begin{cases} \max_{x,\theta} & \overline{V}(\theta) - f(x,\theta) \\ \text{s.t.} & g(x,\theta) \le 0 \\ & Ax + B\theta + d = 0 \\ & \theta \in S \end{cases}$$

this is a convex program !

Result: $\epsilon^{CP}(S) \ge \widehat{V}(\theta) - V^\star(\theta) \ge 0,\ \forall \theta \in S$

Recursive Approximation



Recursive Algorithm

1. Compute $\varepsilon^{CP}(S)$ and let (x_{CP},θ_{CP}) the corresponding worst point

2. If $\varepsilon^{CP}(S)$□ε then for all $i=0,1,\ldots,m$

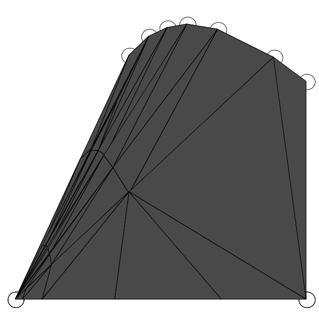
   2.1. Get a new smaller simplex S_i by replacing $\theta_i \leftarrow \theta_{CP}$

   2.2. If S_i is full dimensional, call this algorithm on S_i

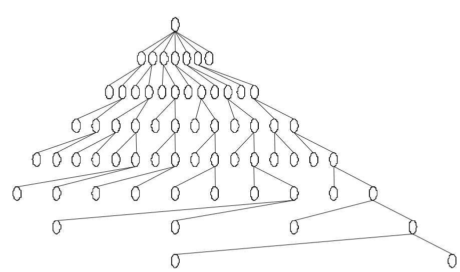
Approximate MP-Convex Solver

- For all simplices $S_{□}$, ..., $S_{N□m□□}$, apply the recursive splitting algorithm to get an approximation with overall error $\leq\varepsilon$

- The approximate multi-parametric solution can be conveniently evaluated via the recursion tree



simplices

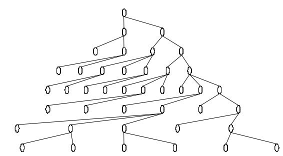


recursion tree

Piecewise Linear Optimal Control laws

- Multi-parametric programming is a useful tool for getting Piecewise Linear (sub)Optimal Control laws

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \le K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \le K_N \end{cases}$$



- Multi-parametric convex nonlinear programming is a promising tool for obtaining robust MPC laws of limited on-line complexity

[1] A. Bemporad and C. Filippi, ``Approximate Multiparametric Convex Programming, submitted CDC 2003.

Paper download: **http://www.dii.unisi.it/~bemporad**

Robust MPC & mp Convex

Robust MPC controllers often are defined as convex optimization problems

$$\begin{array}{rl} \min_x & f(x,\theta) \\ \text{s.t.} & g_i(x,\theta) \le 0 \quad (i = 1,\dots,p) \\ & Ax + B\theta + d = 0 \end{array}$$

⇩

Multiparametric Convex

⇩ ⇩ ⇩

PERFORMANCE	CONSTRAINT HANDLING	STABILITY
$\hat{f}(\theta) \le f^*(\theta) + \epsilon$	$\hat{x}(\theta)$ is feasible ?	?

A case study: Kothare's 96

[1] M.V.Kothare, V. Balakrishna, M.Morari ``Robust Constrained Model Predictive Control using Linear Matrix Inequalities, Automatica 33, 1996

LTV Systems:

$$x_{k+1} = A_k x_k + B_k u_k$$
$$(A_k\ B_k) = \sum \lambda_j (A_j\ B_j)$$

$$V(x) = \min_{P,K} x^T P x$$

$$x^T P x \geq \sum^{\infty} x_k^T Q x_k + u_k^T R u_k$$

Upper bound maximum infinity cost

$$u_k \in U$$
$$x_k \in X$$

Robust constraints

$$u_k = K x_k$$

Linear feedback law

A case study: Kothare's 94 - II

Kothare's optimization problem = SDP

(Conservative solution)

$$\min_{\gamma,Q,Y} \gamma$$

subject to:

MP CONVEX

$$\begin{bmatrix} 1 & x^T \\ x & Q \end{bmatrix} \quad Q > 0$$

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_c^{\frac{1}{2}} & Y^T R_c^{\frac{1}{2}} \\ A_j Q + B_j Y & Q & 0 & 0 \\ QQ_c^{\frac{1}{2}} & 0 & \gamma I & 0 \\ Y^T R_c^{\frac{1}{2}} & 0 & 0 & \gamma I \end{bmatrix} \geq 0 \quad \forall j$$

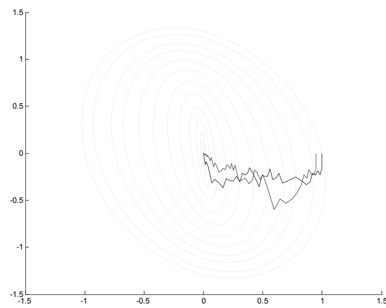
(Unconstrained case)

Properties of the controller

$E_Q = \{x | \, x^T \gamma x \leq \gamma\}$

if $x_k \in E_Q$ then $A_j x_k + B_j K x_k \in E_Q \; \forall j$

$x_k^T P x_k \geq (A_j x_k + B_j K x_k)^T P (A_j x_k + B_j K x_k) \; \forall j$



Stability

Stability proof:

$$x_{k+1}^T P(x_k) x_{k+1} - x_k^T P(x_k) x_k \leq -x_k^T Q x_k - u_k^T R u_k$$
$$\forall \; [A_k B_k] \in \Omega, \; u_k = F(x_k) x_k$$

$$V^*(x_k) \leq x_k^T P(x_k) x_k \leq V^*(x_k) + \epsilon$$

$$x_{k+1}^T P(x_k) x_{k+1} \geq V^*(x_{k+1})$$

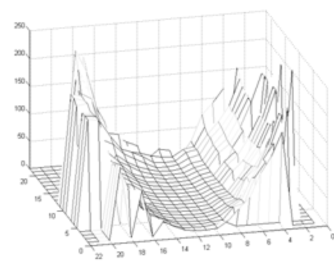
$$\boxed{V^*(x_{k+1}) - V^*(x_k) \leq -x_k^T Q x_k + \epsilon}$$

Examples

$$x_{k+1} = A_k x_k + Bu_k$$

$$(A_k) = \sum \lambda_j (A_j)$$

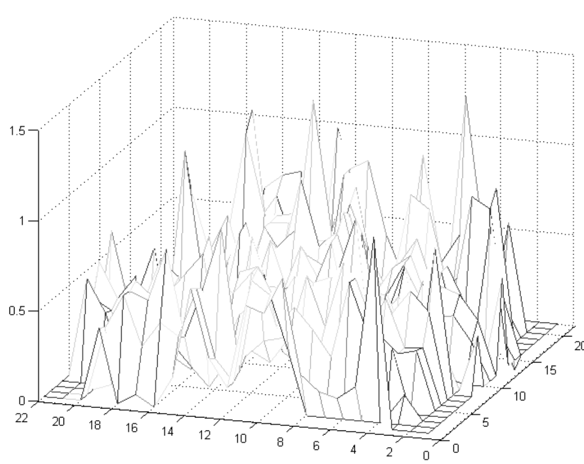
$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}$$



Examples

$\epsilon \leq 1$

1892 regions
10 tree levels



Relative Error bound

Aproximation with maximum relative error

$$\frac{\widehat{V}(\theta) - V(\theta)^*}{V(\theta)^*} \leq \epsilon_R$$

In this case, is not a SDP APROXIMATION

$$\epsilon_R \leq \frac{\epsilon_{CP}}{\max_{\theta_i} V^*(\theta_i)}$$

Where θ_i are the vertices of a given simplex

Stability

DUAL CONTROL

If $\widehat{V}(\theta_{k+1}) \leq \widehat{V}(\theta_k)$ then

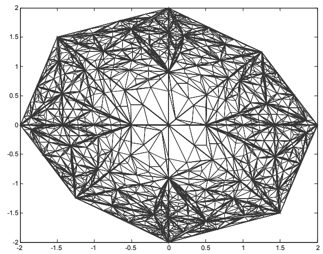
$$x_{k+1} = \widehat{x}(\theta_{k+1})$$

Else

$$x_{k+1} = \widehat{x}(\theta_k)$$

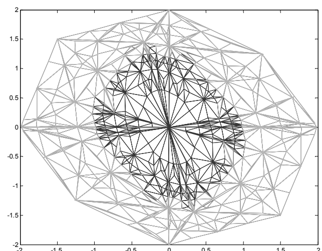
Stability guaranteed for Kothare's properties

Relative error bound



$\epsilon \leq 1$

1892 regions
10 tree levels



$\epsilon \leq 1$ or $\epsilon_R \leq 0.1$

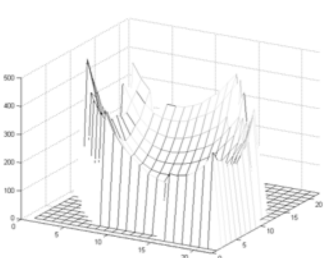
870 regions
7 tree levels

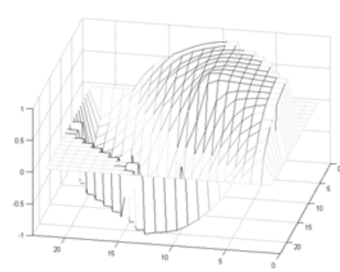
Examples

$$x_{k+1} = A_k x_k + B u_k$$

$$(A_k) = \sum \lambda_j (A_j)$$

$$A_1 = \begin{bmatrix} 0.9 & 0.9 \\ 0 & 0.9 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 0.5 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

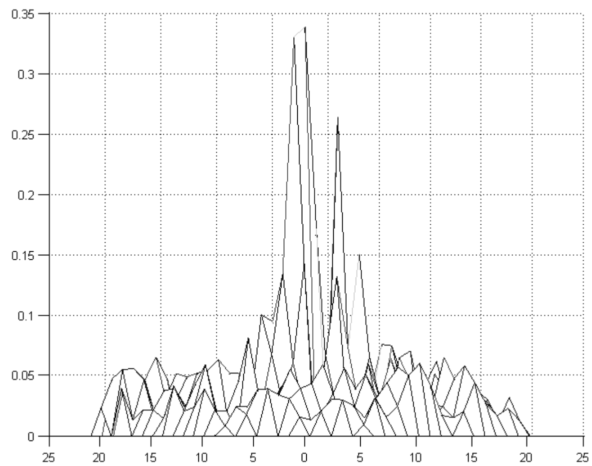




Error bounds

Relative error

$\epsilon \leq 1$ or $\epsilon_R \leq 0.1$



Complexity

nx	nu	xmax	tree	Reg
2	1	1	4	44
2	1	2	6	180
2	1	5	8	500
2	1	10	10	928
3	2	1	5	248
3	2	2	8	3374
3	2	5	12	25512
4	2	1	8	3056
4	2	2	12	3717

Conclusions

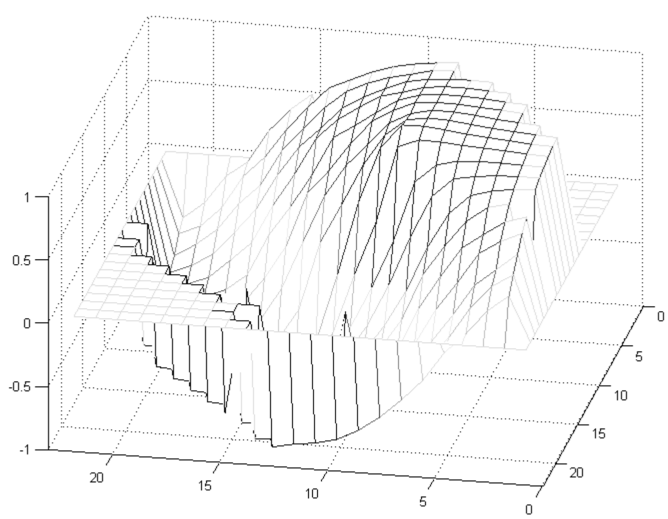
Mp Convex is a tool to take into account
for MPC Robust fast implementation

BUT

Is computing demanding
Case dependent

So if you have:
- A good uncertain model
- Complex control objectives only solvable with a complex robust MPC
- Small sampling time

Take a look on MpConvex



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