

# On solving optimal control problems for switched hybrid nonlinear systems by strong variations algorithms

Mazen Alamir & Sid-Ahmed Attia\*

## Abstract

This paper focuses on the gain in efficiency that may be obtained when using strong variations-like algorithms to solve optimal control problems for switched hybrid nonlinear systems. After a review of existing algorithms, a simple version of a strong variations algorithm is proposed together with some new convergence results. A key features in the proposed algorithm are : 1) the absence of a priori assumptions on the number of switches. 2) The complexity is independent of the state's dimension. 3) The algorithm uses a unified approach for both continuous input and discrete switching strategy. Several examples including a system with 8 states and 7 configurations are used to illustrate the efficiency of the proposed algorithm.

## 1 Introduction

Hybrid systems paradigm is a key issue in process engineering. This is because nowadays systems are becoming more and more complex and a tight modelling of their behaviour at each of their multiple configurations becomes essential to meet the increasingly demanding operating requirements. Moreover, these requirements make essential the use of optimal control strategies. As a result, optimal control of hybrid systems has never been as relevant as it is today.

The computation of optimal control for hybrid systems simultaneously concerns three different issues: modelling, derivation of optimality conditions and design of computable algorithms.

Since the early paper of Witsenhausen [31] that pointed out some features of hybrid systems. Many works have been done to properly define such systems [7, 29, 20]. This mainly amounts to provide sufficient conditions on the system's definition in order to exclude undesirable features like Zeno's behaviour or simultaneous switches. It seems today that there is a broad consensus on what can be considered as a good model for a hybrid system.

Having good models at hand, researchers looked for possible characterizations of optimality enabling an exhaustive search over the whole set of possibilities to be avoided. Such characterizations

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\*Laboratoire d'Automatique de Grenoble. LAG-ENSIEG, CNRS UMR 5528. BP 46, Domaine Universitaire, 28400 Saint Martin d'Hères. e-mail: mazen.alamir@inpg.fr ; Fax: (0)476826388. This work was supported by the European Project CONTROL & COMPUTATION

may be directly obtained by using the Bellman principle via a dynamic programming approach [3, 15] according to which the optimal cost function satisfies the following equation

$$V(x) = \min_u \left[ L(x, u) + V(F(x, u)) \right]$$

where  $L(x, u)$  is the integrand used in the cost function definition while  $F(x, u)$  is the next state resulting from the application of  $u$  starting from  $x$ . Although such characterization is universal and hence directly applies to hybrid systems, its use for systems with high state's dimension is cumbersome since the complexity of the unknown function  $V$  increases exponentially with the state dimension in the general nonlinear case. That is the reason why generalizations of the Pontryagin Maximum Principle (**MP**) have been attempted.

Indeed, the **MP** [23, 10] provides a local characterization of optimality by means of point-wise (in time) necessary extremality conditions. Despite its local and only necessary character, the **MP**-based characterizations proved to be efficient in finding "good" solutions in the context of classical controlled systems. However, the **MP** is not as universal as the Bellman principle and its generalization to hybrid systems is not self evident [29, 25, 30, 20]. Roughly speaking, the derivation of maximum principle for hybrid systems amounts to a multiple use of constrained versions of the classical **MP**, the constraints being used to impose the switches-related conditions. This results in a much more complicated necessary transversality conditions.

As for concrete computational algorithms, several attempts have been done and the subject is concentrating much effort since it seems to be one of the burning issues for the control engineering community that is concerned with hybrid systems. An excellent almost exhaustive recent survey of existing attempts can be found in [33] with a particular classification scheme. Here, the basic methodologies are briefly described in order to underline the novelty of the proposed approach.

In [14], the universality of the dynamic programming principle is used to compute the optimal cost function  $V(x)$  mentioned above. The complexity is maintained at a low level by projecting the computed cost function at the end of each iteration on a low dimensional functional basis. This projection is done under precision-related constraint yielding a controlled approximation. For truly complicated optimal cost function  $V(x)$  however, and despite this clever trick, the methodology remains limited to system with low number of states. The dynamic programming paradigm is also invoked in [8, 9] using level sets and behavioural programming with roughly the same unfortunate explosion of complexity with the state dimension.

Another approach that gains increasing number of followers everyday is the one based on Mixed-Integer Logical Dynamics (**MILD**) formulation where basically linear dynamics are used with extended state containing the continuous and binary variables [6, 4]. This strategy is based on the fact that for piece-wise affine problems, the optimal solution is necessarily on one corner of some high dimensional polygon. The approach is very attractive since systematic tools begin to appear enabling an easy implementation and because of the wide class of hybrid features and constraint that it may conceptually tackle. however, when it is used to solve truly nonlinear problems, the dimension of the decision variable increases exponentially with the state dimension and the solution needs branch & bound -like approaches to explore the huge combinatorial set of possibilities. To

reduce complexity, one may be tempted to reduce the precision with which the piece-wise affine approximation is achieved, unfortunately, in this case, the problem being solved may differ from the original one.

In ([13, 12, 32], a two stage approach has been proposed. In the first stage, the total number of switches is a priori fixed as well as the sequence of active subsystems. By doing this, the cost function is only function of the switching instants. Therefore, finding the optimal switching instants for the given number of switches and the sequence of active subsystems is a classical nonlinear programming problem. In the second stage, the a priori given data, namely, the number of switches and the sequence of active subsystems are updated to improve the optimal solution that would be obtained by the first stage. Unfortunately, only the first stage is really investigated. But by doing so, the *only* real difficulty in hybrid systems framework is eluded, namely, how to tackle the combinatorics associated to the choice of the sequence of active configurations.

Using roughly the same two-stage philosophy, master/slave algorithms have been developed for linear systems under infinite horizon interval cost functions [5]. In particular, the second stages is solved using the **MILD** tools invoked above since only linear systems are studied.

The work by [27, 28] uses the same two-stage structure but uses explicitly the maximum-principle in deriving the updated value of the switching instants. In [18] optimizing the switching instants when the number of switches and the sequence of active configurations is given is achieved using either constrained nonlinear programming of the maximum principle.

In this paper, it is shown that as long as switched nonlinear hybrid systems with only external switching controls are concerned, strong variations algorithms enables a unified approach that iterates on both continuous and logical variables and avoid the use of a priori assumptions on the number of switches or the sequence of active configurations.

Strong variations algorithms are based on a leading paper of R. V. Gamkrelidze [11] where it has been shown that variations in the control profiles that are small in the  $L_2$  norm but not in the extremum norm may be used in the iterations when looking for an optimal control profile. This makes possible to imagine algorithms for optimal control problems in which the admissible control sets are not necessarily convex [19, 21, 16, 2]. In particular, discrete or boolean sets can be naturally handled by such algorithms. This is of a crucial interest for our concern. The use of strong variations algorithms to solve hybrid optimal control problems has already been suggested in [1]. The present papers confirms the conjecture stated there about the efficiency that such algorithms may have in tackling this problem.

The paper is organized as follows: First the problem in question is clearly stated in section 2 and re-formulated in a rather standard form in section 3. In section 4 a simple algorithm is proposed with some convergence results. The algorithm may be viewed as a free-disturbance version of an existing algorithm that has been proposed in [2] to solve nonlinear differential games over non convex sets in the context of robust model predictive control of batch processes. However, the fact that only the free-uncertainty case is considered here, new stronger convergence results are derived here that might not hold for the min-max version of the algorithm given in [2]. Finally, in section

5, the algorithm is applied on several examples of switched hybrid systems to assess its efficiency in solving the optimal control problem under concern. It is particularly shown that for relatively large state's dimension, no combinatoric effects appear nor tree search-like task is needed. The paper ends with some concluding remarks suggesting some ideas for further investigations.

## 2 Problem statement

Let us consider a nonlinear system that may be in  $Q$  different configurations. Let

$$q \in \underline{Q} := \{1, \dots, Q\}$$

be a discrete state variable used to designate the active configuration of the system. It is assumed that both the continuous state of and the control input, denoted respectively by  $x \in \mathbb{R}^n$  and  $v \in \mathcal{V} \subset \mathbb{R}^m$  are uniquely defined whatever is the active configuration  $q$ . When the system is at configuration  $q \in \underline{Q}$ , its dynamics is described by

$$\dot{x} = f_q(x, v) \quad ; \quad x \in \mathbb{R}^n \quad ; \quad v \in \mathcal{V} \subset \mathbb{R}^m \quad ; \quad q \in \underline{Q} \quad (1)$$

where  $\mathcal{V} \subset \mathbb{R}^m$  is some set of admissible controls.

To complete the system's definition, one has to describe how the system switches from one configuration to another. In the general hybrid systems framework, this may be induced by external inputs (controlled switches) and/or when the state crosses some switching boundary (autonomous switches). In the present paper, interest is focused on hybrid systems with only controlled switches. Furthermore, the jump is assumed to be completely free in the sense that whatever is the present configuration, the system may jump to any other configuration. The aim of this paper is to propose a heuristic algorithm with appropriate convergence properties that provides a *good sub-optimal* switching strategy  $q^*(\cdot)$  between configurations and a corresponding *good sub-optimal* control  $v^*(\cdot)$  such that the following functional is minimized

$$J(x_0, q(\cdot), v(\cdot)) := \int_0^T L_{q(\tau)}(x(\tau), v(\tau)) d\tau \quad (2)$$

under the following free-chattering constraint

$$\forall (t_1, t_2) \in [0, T]^2 \quad , \quad \left\{ q^*(t_1) \neq q^*(t_2) \right\} \quad \Rightarrow \quad |t_2 - t_1| > dt_{min} > 0 \quad (3)$$

where for all  $q \in \underline{Q}$ ,  $L_q : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$  is some penalty function depending on the system's configuration while  $dt_{min}$  stands for some minimum time delay between two successive controlled switches.

**Remark 1** *Since the controlled switches are completely free, the constraint (3) is necessary to avoid infinite number of switches in finite time. Apart from this theoretical need, this condition is useful to obtain a realistic control strategy that is necessarily sampled.*

Although the class of systems that meet the assumptions above seems to be restrictive compared to what a general hybrid system might be, it is worth noting that most of the existing algorithms

are restricted to such systems. Anyway, this paper aims to draw one's attention to the power of a particular family of algorithms in avoiding combinatoric effects. Restricting this first attempt to the simple class of hybrid systems defined above makes it possible to easily grasp the heart of the approach and hence incite its generalization to more complex hybrid systems.

In the following section, it is shown that the above problem can be put in a general form to yield a classical nonlinear optimal control problem over non convex sets of admissible controls. This is a first step to apply the proposed algorithm that is dedicated to such general classical although non convex optimal control problems.

### 3 Problem's re-formulation

The system's dynamics can be clearly re-written as follows

$$\dot{x} = \sum_{q \in \underline{Q}} \alpha_q f_q(x, v) \quad (4)$$

provided that for all  $q^0 \in \underline{Q}$ ,  $\alpha_{q^0}$  is equal to 1 if  $q^0$  is the "active" configuration and  $\alpha_{q^0} = 0$  otherwise. More precisely, the  $\alpha_q$ 's have to meet the following requirements

$$\forall q \in \underline{Q} \quad , \quad \alpha_q \in \{0, 1\} \quad \text{and} \quad \sum_{q \in \underline{Q}} \alpha_q = 1 \quad (5)$$

namely, one and only one of the  $\alpha_q$ 's is equal to 1 while the others are all 0's. In what follows, the constraint (5) is formally written as follows

$$\alpha = (\alpha_1 \quad \dots \quad \alpha_Q)^T \in \mathcal{A} \quad (6)$$

note that  $\mathcal{A} \subset \{0, 1\}^Q$  is a discrete subset with  $card(\mathcal{A}) = Q$ . with this notation, equation (4) becomes

$$\dot{x} = F(x, v, \alpha) := \sum_{q \in \underline{Q}} \alpha_q f_q(x, v) \quad ; \quad v \in \mathcal{V} \quad \text{and} \quad \alpha \in \mathcal{A} \quad (7)$$

and by concatenating  $(v, \alpha)$  in a single control vector

$$u := (v, \alpha) \in \mathcal{V} \times \mathcal{A}$$

it comes that the class of hybrid systems under study may be written in the following general form

$$\dot{x} = f(x, u) \quad ; \quad u \in U := \mathcal{V} \times \mathcal{A} \subset \mathbb{R}^m \times \{0, 1\}^Q \quad (8)$$

with a non convex set  $U$  of admissible controls.

**Remark 2** *It is thanks to the free-controlled switches assumption that makes it possible to view the  $\alpha$  in (7) as a rather classical control even though it belongs to a non convex set  $\mathcal{A}$ . Again, this is a serious restriction even though generalization may be quite easily considered in further studies.*

Similarly, the cost function may be rewritten as follows

$$J(x_0, u) := \int_0^T \sum_{q=1}^Q \alpha_q(\tau) L_q(x(\tau), v(\tau)) d\tau =: \int_0^T L(x(\tau), u(\tau)) d\tau \quad ; \quad u := (v, \alpha) \quad (9)$$

where for all  $x \in \mathbb{R}^n$  and all  $u = (v, \alpha) \in U$ , the following definition is used

$$\forall u = (v, \alpha) \in U \quad , \quad L(x, u) := \sum_{q=1}^Q \alpha_q L_q(x, v) \quad (10)$$

Again, the cost function (10) is rather classical. Moreover, if for all  $q \in \underline{Q}$ ,  $L_q$  is  $C^r$  in its arguments, for some  $r \in \mathbb{N}$  then  $L$  is  $C^r$  in its arguments.

In the following section, a simple algorithm is proposed to find "good" sub-optimal solutions  $u(\cdot)$  to the problem of minimizing (9) under (8) with controls that belong to the non convex set  $U$ .

## 4 The proposed algorithm

In this section, a general purpose algorithm is given to solve optimal control problems in which the control input may be restricted to possibly non convex admissible sets. Recall that the algorithm proposed hereafter is a free-disturbance version of the algorithm proposed by [2]. The latter enables min/max differential games to be solved. However, in [2], the convergence results are quite poor because of the complexity introduced by the min/max related features. Here, the algorithm is adapted to the free-disturbance case making possible the derivation of stronger convergence results. By the same, the success of the proposed algorithm (see the examples of section 5) suggests that the algorithm in [2] may be used to solve min/max hybrid optimal control problems. For the time being, let us concentrate on the free-uncertainties optimal control problem for switched hybrid systems since even this issue is still quite open.

Consider the following constrained optimization problem

$$\text{Minimize} \quad J(u) = \int_{t_0}^{t_f} L(x(t), u(t), t) dt \quad (11)$$

subject to the system equations

$$\dot{x}(t) = f(x(t), u(t), t) \quad ; \quad x(t_0) = x_0 \quad (12)$$

and the control constraint

$$u(t) \in U \quad \forall t \in [t_0, t_f] \quad (13)$$

where  $U$  is not necessarily convex. Assume that the following assumption holds

ASSUMPTION 1

- The functions  $f$  and  $L$  are twice continuously differentiable. (Note that for the original problem this can be guaranteed by assuming twice continuous differentiability of  $f_q$  and  $L_q$  for all  $q \in \underline{Q}$ ).
- There is a positive real  $M > 0$  such that for all admissible control  $u$ , the corresponding state trajectory  $x(t; t_0; x_0, u)$  satisfies the following inequality

$$\forall t \in [t_0, t_f] \quad , \quad \| x(t; t_0; x_0, u) \| \leq M \quad (14)$$

To solve the above problem for non convex admissible sets  $U$ , several existing algorithms may be used such as the ones given in [16, 19, 21]. However, these strong variations algorithms are rather tedious to properly encode. The proposed algorithm, based on a variable penalty technique is very simple while enabling convergence results to be derived.

From the hybrid systems optimal control problem viewpoint, the algorithms cited above would probably give similar results than the ones we obtain. Anyway, to the best of our knowledge, there is no commercial nor free approved software implementing the underlying strong variations techniques. One of the main contribution of the present paper may be to incite works that would fill the gap.

## 4.1 The maximum principle

The algorithm proposed here is based on the well known classical maximum principle [22] that is recalled hereafter. Let  $u^*(\cdot)$  be an optimal control profile and let  $x^*(\cdot)$  be the corresponding trajectory defined on  $[t_0, t_f]$ . Then a necessary condition is the existence of a corresponding co-state  $\lambda^*(\cdot)$  defined over  $[t_0, t_f]$  such that the following conditions hold

1.  $\lambda^*$  satisfies the following differential equations over  $[t_0, t_f]$

$$\dot{\lambda}^*(t) = -H_x(x^*(t), u^*(t), \lambda^*(t), t) \quad ; \quad \lambda^*(t_f) = 0 \quad (15)$$

where  $H$  is the Hamiltonian of the problem defined by

$$H(x, u, \lambda, t) = L(x, u, t) + \lambda^T f(x, u, t) \quad (16)$$

2. The optimal solution  $u$  minimizes the Hamiltonian over the admissible set  $U$ , that is

$$H(x^*(t), u^*(t), \lambda^*(t), t) = \min_{u \in U} H(x^*(t), u, \lambda^*(t), t) \quad (17)$$

where in the above,  $x^*(\cdot)$  stands for the solution of (12) under the optimal control profile  $u^*$ .

## 4.2 The algorithm

It is the characterization (17) of the optimal solution that will be used in our algorithm. It is clear that the maximum principle is an infinite dimensional sufficient condition in the sense that it must hold for all  $t \in [t_0, t_f]$ . An implementable algorithm, however, must be finite dimensional and all convergence investigations are to be done on some finite dimensional approximation of the optimal control problem.

One natural way to define a finite dimensional version of the above problem is to define a sufficiently dense grid

$$t_0 = t_1 < t_2 < \dots < t_N = t_f \quad ; \quad t_{k+1} = t_k + h \quad ; \quad h = \frac{t_f - t_0}{N - 1} \quad (18)$$

over the time interval  $[t_0, t_f]$  in order to approximate the solutions of (12) and (15) with a sufficiently high precision using a second order integration method. More specifically, given an  $(N - 1)m$ -dimensional vector  $\bar{u}$

$$\bar{u} = [\bar{u}^T(1), \dots, \bar{u}^T(N - 1)] \in \bar{U} := U \times \dots \times U \subset \mathfrak{R}^{(N-1)m}$$

we shall identify the vector  $\bar{u} \in \mathfrak{R}^{(N-1)m}$  to the corresponding piece-wise constant control defined over  $[t_0, t_f]$  by

$$u(t_k + \tau) = \bar{u}(k) \quad ; \quad k = 1 \dots N \quad ; \quad \tau \in [0, h[ \quad (19)$$

Furthermore, the following finite dimensional approximations

$$\bar{x} := [\bar{x}^T(1), \dots, \bar{x}^T(N)]^T \in \mathfrak{R}^{Nn} \quad ; \quad \bar{\lambda} := [\bar{\lambda}^T(1), \dots, \bar{\lambda}^T(N)]^T \in \mathfrak{R}^{Nn}$$

are defined by

$$\bar{x}(k + 1) := \bar{x}(k) + \frac{h}{2} \left[ f(\bar{x}(k), \bar{u}(k), t_k) + f(\bar{x}(k) + hf(\bar{x}(k), \bar{u}(k), t_{k+1})) \right] \quad ; \quad \bar{x}(1) = x_0 \quad (20)$$

$$\begin{aligned} \bar{\lambda}(k - 1) := & \bar{\lambda}(k) + \frac{h}{2} \left[ H_x \left( \bar{x}(k - 1), \bar{u}(k - 1), \bar{\lambda}(k) + hf(\bar{x}(k), \bar{u}(k - 1), \bar{\lambda}(k), t_k), t_{k-1}) \right) + \right. \\ & \left. H_x(\bar{x}(k), \bar{u}(k - 1), \bar{\lambda}(k), t_k) \right] \quad ; \quad \bar{\lambda}(N) = 0 \end{aligned} \quad (21)$$

With the discretization scheme defined above, an approximate value of the cost function can be obtained for all  $\bar{u} \in \bar{U}$  using  $\bar{J}(\bar{u})$  given by

$$\bar{J}(\bar{u}) = h \sum_{k=1}^{N-1} L(\bar{x}(k), \bar{u}(k), t_k) \quad (22)$$

Given the above definitions, consider the following algorithm



ALGORITHM

- **Step 0:** Fix some small  $\epsilon_u > 0, \epsilon_J > 0$ , some integer  $i_{max}$  and two reals  $d\mu > 0$  and  $\gamma > 1$ . Choose  $\mu^0 \geq 0$  and some initial admissible guess  $\bar{u}^0 \in \bar{U}$ .
- **Step 1:** Compute  $\bar{x}^0$  solution of (20) with  $\bar{u} = \bar{u}^0$ , let  $i = 1$ ,
- **Step 2:** Compute  $\bar{\lambda}^{i-1}$  solution of (21) with  $\bar{u}^{i-1}$  and  $\bar{x}^{i-1}$  already computed,
- **Step 3:** Compute  $\bar{u}^i$  and  $\bar{x}^i$  such that
  - $\bar{x}^i$  is solution of (20) with  $\bar{u} = \bar{u}^i$  such that
  - $\bar{u}^i(k) := \text{Arg} \min_{u \in U} \left[ H(\bar{x}^i(k), u, \bar{\lambda}^{i-1}(k), t_k) + \mu^{i-1} \|u - \bar{u}^{i-1}(k)\|^2 \right]$ ,
- **Step 4:** **If**  $\left( \bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$  **and**  $\left( \|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u \right)$  **then** let  $\mu^{i-1} = \max\left(\mu^{i-1} + d\mu, \gamma\mu^{i-1}\right)$  and return to **Step 3**,
- **Step 5:** **If**  $\left( \bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$  **then**  $\mu^i = \max\left(0, \min(\mu^{i-1} - d\mu, \mu^{i-1}/\gamma)\right)$ ,
- **Step 6:** **If**  $\left( \|\bar{u}^i - \bar{u}^{i-1}\| \leq \epsilon_u \right)$  **and**  $\left( i \geq i_{max} \right)$  **Then stop else** let  $i = i + 1$  and return to **Step 2**.

This algorithm may be understood in the light of the following remarks. Rigorous convergence results are given in the following subsection and the corresponding proofs are given in the appendix.

- ✓ The algorithm begins with some initial guess  $\bar{u}^0$ . The key problem is that such an arbitrary choice is generically incompatible with the maximum principle's necessary condition (17).
- ✓ **Step 3** produces then a *chattering* behaviour between the proceeding step's guess  $\bar{u}^{i-1}$  and the one that minimizes the corresponding *modified* Hamiltonian in the construction of which,  $\bar{x}^{i-1}$  and  $\bar{\lambda}^{i-1}$  are used.
- ✓ The modification of the Hamiltonian by the penalty term  $\mu^{i-1}\|u - \bar{u}^{i-1}(k)\|^2$  enables the successive iterations to be stabilized. Indeed, when  $\mu^{i-1}$  is high,  $u$  will be different from  $\bar{u}^{i-1}$  only if it dramatically decreases  $H$ .
- ✓ Note that **Step 4** leads to high values of  $\mu^i$  if the algorithm remains between **Step 3** and **Step 4**. These high values stabilize the iterations as explained above and force the algorithm to leave this loop after a finite number of iterations (see corollary 2 hereafter). The aim of **Step 5** is then to reduce  $\mu^i$  that may have been temporary increased (in **Step 4**) in order to avoid some local difficulties. This is because, from the maximum principle viewpoint, the minimization in **Step 3** has sense only for  $\mu^{i-1} = 0$ .
- ✓ The condition  $\left( \bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$  **and**  $\left( \|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u \right)$  in **Step 4** is to be interpreted as follows: **If**  $\bar{u}^i$  fails to decrease significantly the cost function, that is  $\left( \bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$  **and if** there is still evolution margin (that is  $\left( \|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u \right)$ ) **then** return to **Step 3** with the new value of  $\mu^i$ .

- ✓ Note that both additive and multiplicative factors are used to decrease  $\mu$ . Indeed, the multiplicative factor guarantees a fast decrease that may be necessary if very high values are needed to stabilise the iterations. The additive term enables to avoid asymptotic decrease of  $\mu$  that would never exactly reach the key value 0 (see point 2. of corollary 2 hereafter).

### 4.3 Some convergence results

The following proposition is the key convergence result. It concerns the behavior of the approximate cost function values at successive iterations

**PROPOSITION 1 [Behavior of successive cost function values]**

*there are positive reals  $r > 0$  and  $\sigma > 0$  such that, for sufficiently small step size  $h$ , the solutions of successive iterations satisfy the following inequality*

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h(r - \mu^{i-1}) \sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 + \sigma \quad (23)$$

**Proof** See appendix A.1.

Note that inequality (23) clearly shows that if

$$\sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 \neq 0$$

then one can always decrease the cost function by letting  $\mu^{i-1}$  taking sufficiently high values. But this is exactly what is **Step 4** dedicated to. Now since  $J^i$  cannot indefinitely decrease, the quantity  $\|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|$  must vanish.

From proposition 1, we can easily derive the following corollaries

**COROLLARY 1 [Convergence of the cost function]**

*Let  $(\bar{x}^i, \bar{u}^i)$  be a sequence generated by the algorithm. The corresponding sequence of cost values  $\bar{J}(\bar{u}^i)$  are monotonically decreasing. Moreover, if  $i_{max} = \infty$ , the infinite sequence  $\bar{J}(\bar{u}^i)$  is convergent. ♠*

**Proof** See appendix A.2.

Finally, the following is the basic result of the present paper

**COROLLARY 2 [Convergence of the control sequence]**

*Let  $(\bar{x}^i, \bar{u}^i)$  be a sequence generated by the algorithm.*

1. *Suppose that  $i_{max} = \infty$  in order to generate an infinite sequence  $\bar{u}^i \in \bar{U}$ . There is an integer  $\bar{i}$  such that*

$$\forall i \geq \bar{i} \quad : \quad \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\| = 0 \quad ; \quad k = 1 \dots N - 1 \quad (24)$$

Moreover, if 0 is an accumulation point for the sequence  $\mu^{i-1}$  then there is an accumulation point  $\bar{u}^*$  of the sequence  $\bar{u}^i$  that satisfies the maximum principle on the grid points, that is

$$H(\bar{x}^*(k), \bar{u}^*(k), \bar{\lambda}^*(k), t_k) \leq H(\bar{x}^*(k), \bar{u}, \bar{\lambda}^*(k), t_k) \quad ; \quad \text{for all } \bar{u} \in \bar{U} \quad (25)$$

2. In particular, if for some  $\bar{i}$ , one has  $\bar{u}^{\bar{i}} = \bar{u}^{\bar{i}-1} =: \bar{u}^*$  with  $\mu^{\bar{i}-1} = 0$  then the control  $\bar{u}^*$  satisfies the maximum principle (25) at the grid points.
3. In the case where  $i_{max}$  is finite, the algorithm stops after a finite number of iterations. Namely, it cannot be trapped by the loop in **(Step 3)**-(**Step 4**)

**Proof** See appendix A.3.

## 5 Illustrative examples

In this section, some examples are used to illustrate the efficiency of the proposed algorithm. Experiments have been conducted using a FORTRAN 90-compiler on a PENTIUM III-600Mhz Personal Computer.

### 5.1 Example 1 [17, 24, 26, 16]

Consider the switched nonlinear hybrid system given by

$$\dot{x} = f_q(x) := \left( \begin{array}{c} x_2 \\ -x_1 + (1.4 - 0.14x_2^2)x_2 + 4\left(\frac{2(q-1)}{Q-1} - 1\right) \end{array} \right) \quad ; \quad q \in \{1, \dots, Q\} \quad (26)$$

having  $Q \in \mathbb{N}$  distinct configurations in each of which the system is autonomous, namely, no continuous control input  $v$  appears in the general form (1). The optimal control problem is defined by taking

$$L_q(x) = x_1^2 + \left[ \frac{2(q-1)}{Q-1} - 1 \right]^2 \quad ; \quad T = 2.5 \text{ s} \quad (27)$$

It can be easily seen that when  $Q$  tends to infinity, the corresponding hybrid optimal control problems tends to the following classical one

$$\min_u \int_0^{2.5} (x_1^2(t) + u^2(t)) dt \quad \text{under} \quad \dot{x} = \left( \begin{array}{c} x_2 \\ -x_1 + (1.4 - 0.14x_2^2)x_2 + 4u \end{array} \right) \quad \text{and} \quad |u(t)| \leq 1 \quad (28)$$

This optimal control problem has been studied in [17, 24, 26, 16] and the optimal value is known to be equal to 42.8 that can be viewed as a lower bound of the optimal solution of the hybrid optimal control problem since for all  $Q$  and all  $q \in \underline{Q}$ , one clearly has

$$-1 \leq \frac{2(q-1)}{Q-1} - 1 \leq 1$$

This enables the quality of the sub-optimal solutions to be appreciated. This is the main interest of this example.

Iterations	Cost function	$\mu$	$\ \bar{u}^i - \bar{u}^{i-1}\ _\infty$
1	146.83	10	20.0
2	47.43	8.33	11.0
3	43.47	6.94	3.0
4	43.24	5.79	1.0
5	43.16	4.79	1.0
6	43.10	3.79	1.0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	42.99	0.79	1.0
10	42.99	0.00	0.00

Table 1: Convergence results for example 1. (Execution time  $\approx 0.2$  s)

The hybrid optimal control problem corresponding to  $Q = 21$  has been solved using  $N = 201$  in (18), that is  $h = 2.5/200$  s. Consequently, the corresponding decision variable is basically of dimension

$$21^{200}$$

The parameters in question in step 0 of the algorithm are as follows

$$\epsilon_u = \epsilon_J = 0.001 \quad ; \quad d\mu = 1 \quad ; \quad \gamma = 1.2 \quad ; \quad \mu_0 = 10$$

while the initial guess  $\bar{u}^0(\cdot) \equiv -1$  has been used.

The convergence results of the algorithm are shown on table 1 with the obvious notation

$$u = q$$

in accordance with the notations used in the problem's reformulation since there is no continuous input (see section 3).

Note the monotonic decrease of the cost function to a value that is quite close to the lower bound 42.8 mentioned above. Higher values of  $Q$  lead to closer values of the sub-optimal cost function. Note also that according to point 2. of corollary 2, the iterations stop at an extremal point satisfying the maximum principle at the grid points since for  $i = 10$ , one obtains  $\mu^i = 0$  and  $\bar{u}^i - \bar{u}^{i-1} = 0$ .

The switching strategies obtained at different iterations as well as the initial and the optimal state trajectories are given on figure 1. Note that during the optimal scenario, about 13 controlled switches take place. This suggests that strategies based on the use of an a-priori given number of switches have to explore a  $21^{13}$ -dimensional tree corresponding to all the possible sequences of active configurations for each choice of the 13 switching instants.

## 5.2 Example 2 [32]

In [32], an algorithm has been proposed to solve optimal control problems on switched hybrid systems. Several examples have been used to illustrate the proposed algorithm. In particular, the

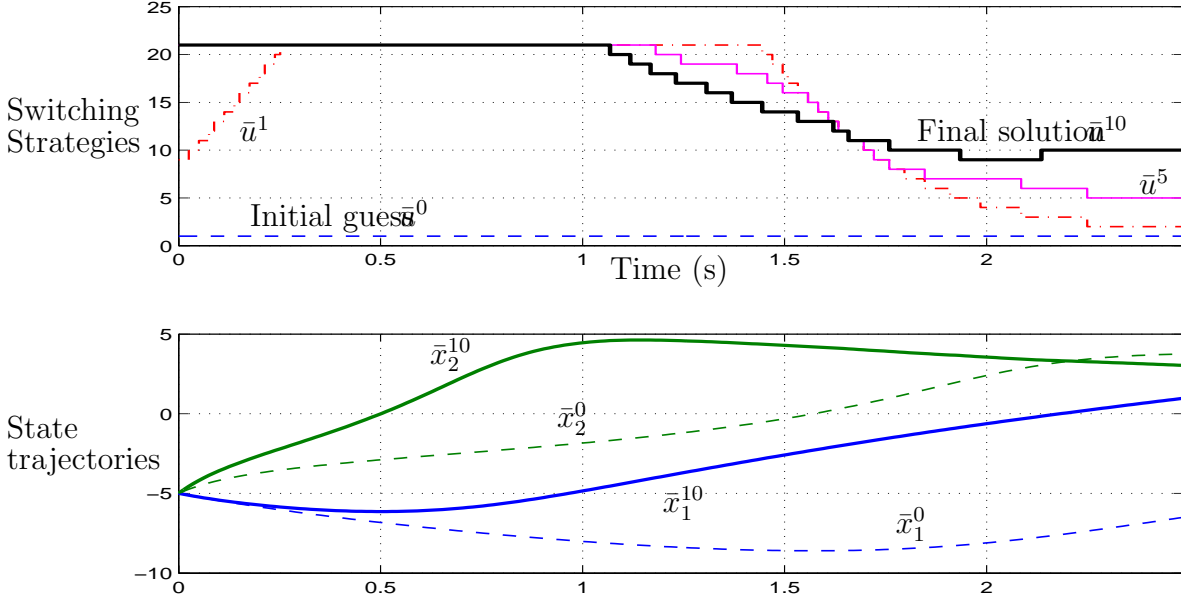


Figure 1: switching strategies and state trajectories at different iterations (Example 1)

following bilinear system has been considered (the other examples in [32] concern linear systems)

$$\begin{aligned}
 \text{subsystem 1: } & \begin{cases} \dot{x}_1 = -x_1 + 2x_1v \\ \dot{x}_2 = x_2 + x_2v \end{cases} \\
 \text{subsystem 2: } & \begin{cases} \dot{x}_1 = x_1 - 3x_1v \\ \dot{x}_2 = 2x_2 - 2x_2v \end{cases} \\
 \text{subsystem 3: } & \begin{cases} \dot{x}_1 = 2x_1 + x_1v \\ \dot{x}_2 = -x_2 + 3x_2v \end{cases}
 \end{aligned}$$

with the following cost function

$$J = \frac{1}{2}(x_1(T) - 2)^2 + \frac{1}{2}(x_2(T) - 2)^2 + \frac{1}{2} \int_0^T \left[ (x_1(t) - 2)^2 + (x_2(t) - 2)^2 + v^2(t) \right] dt \quad ; \quad T = 3 \quad (29)$$

with the initial state  $x(0) = (1 \ 1)^T$ .

A sub-optimal solution of the hybrid optimal control problem above has been obtained in [32] based on the a priori assumption of 2 switching instants. The following sub-optimal cost has been obtained

The sub-optimal  $J$  achieved by [32] is 3.625

The sub-optimal switching strategy is characterized by one switch at instant  $t^* = 1.7244$  taking the system from configuration  $q = 2$  to configuration  $q = 3$ . This can be seen on the dotted lines plots of figure 2.

Let us see how the problem above can be put in the framework of the present paper. The

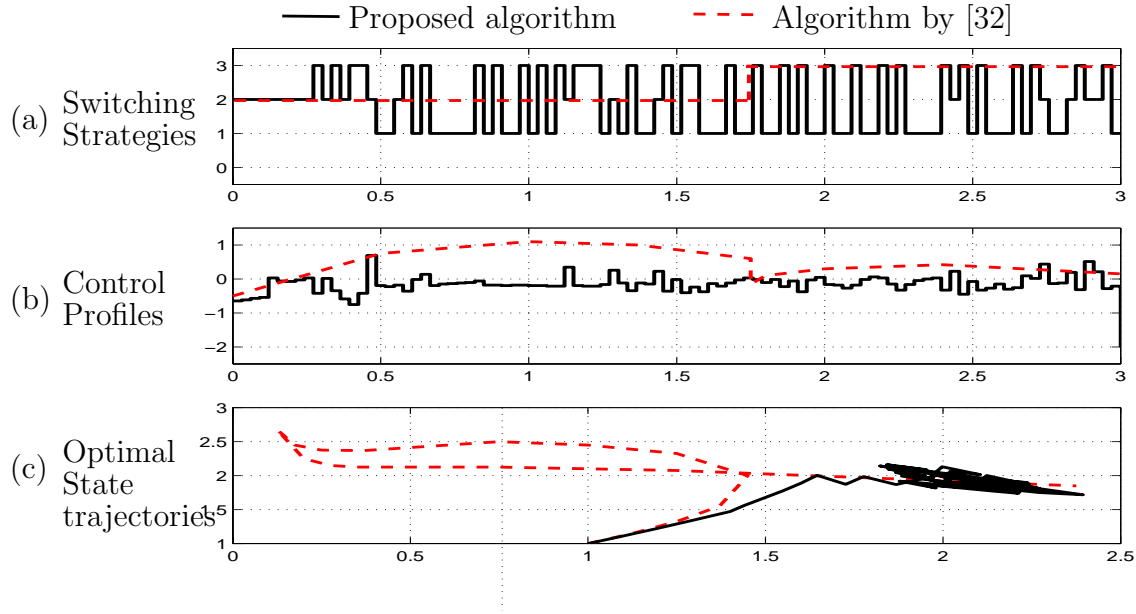


Figure 2: switching strategies, control profiles and state trajectories for example 2. (Dotted lines - -) the sub-optimal solution achieved by [32]. (Solid lines -) The sub-optimal solution achieved by the proposed algorithm.

system admits three different configurations ( $Q = 3$ ) and can be put in the general form (1) by using the following notations

$$f_1(x, v) = \begin{pmatrix} -x_1 + 2x_1v \\ x_2 + x_2v \end{pmatrix} ; f_2(x, v) = \begin{pmatrix} x_1 - 3x_1v \\ 2x_2 - 2x_2v \end{pmatrix} ; f_3(x, v) = \begin{pmatrix} 2x_1 + x_1v \\ -x_2 + 3x_2v \end{pmatrix} \quad (30)$$

By the same, the cost function can be put in the standard form (2) by using the following straightforward definition of  $L_q$  for  $q \in \{1, 2, 3\}$

$$L_q(x, v) := \frac{1}{2} \left[ (x_1 - 2)^2 + (x_2 - 2)^2 + v^2(t) \right] + \frac{\partial \Psi}{\partial x}(x) \cdot f_q(x, v) \quad ; \quad \Psi(x) = \frac{1}{2}(x_1 - 2)^2 + \frac{1}{2}(x_2 - 2)^2$$

where the final penalty in (29) has been transformed into an integral form.

The proposed algorithm has been successfully used to solve the corresponding hybrid optimal control problem using the following parameters

$$\epsilon_u = \epsilon_J = 0.001 \quad ; \quad d\mu = 0.5 \quad ; \quad \gamma = 1.5 \quad ; \quad \mu_0 = 10$$

That is, only  $\gamma$  and  $d\mu$  have been slightly modified comparing to the set of values used to solve example 1. Note that without this changes, the algorithm obtained still good solutions. However, the values above seemed to give the better results.

The optimal cost achieved by the algorithm is given by

$$\hat{J}_{\text{proposed algorithm}} = 0.3742$$

to be compared to 3.625 obtained in [32].

The convergence history for example 2 is shown on figure 3. Indeed, on this figure, one may see the evolution of the cost function  $J^i$ , the penalty parameter  $\mu^i$  and the key indicator  $\|\bar{u}^i - \bar{u}^{i-1}\|_\infty$ .

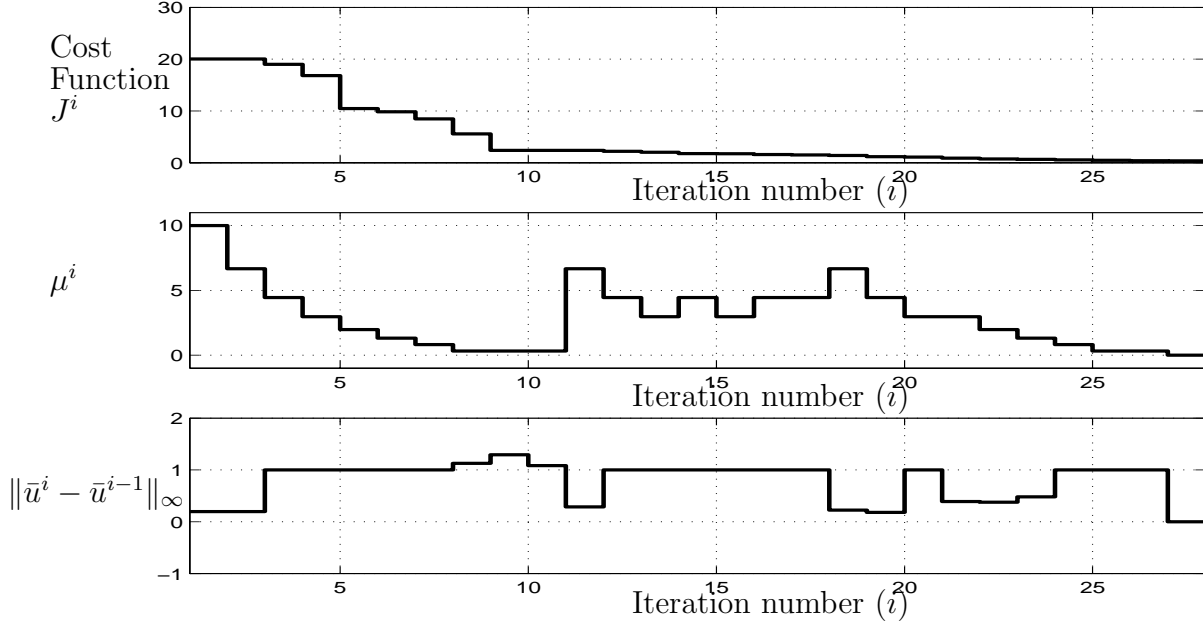


Figure 3: Convergence results for example 2 (Execution time  $\approx 1s$ )

careful look on figure 3 enables the following fact to be underlined

- ✓ At iteration 28, the condition required by corollary 2 are satisfied, namely

$$\mu^i = 0 \quad ; \quad \|\bar{u}^i - \bar{u}^{i-1}\|_\infty = 0$$

therefore, according to corollary 2, the sequence so obtained satisfies the maximum principle at the grid points.

- ✓ The evolution of  $\mu^i$  is not monotonic, the dynamics of this penalty is a key issue in the convergence of the iterations.

As for the optimal trajectories, they can be viewed on figure 2 where the optimal switching strategy, the optimal control input and the optimal states trajectories (in the phase plane) are given in solid lines. These are to be compared with the dotted lines that summarize the results of [32]. Figure 2 suggests the following comments

- ✓ Note the high number of switches "asked by the proposed algorithm". This is because, in the absence of switches, keeping the state near the point (2, 2) as implied by the cost function would need a control  $v$  that is far from 0 whatever is the configuration being used and this is penalized by the squared penalty term on the control  $v$ . That is why a 2-switches strategy necessarily corresponds to a high cost.

However, by using cleverly and intensively the switches, the algorithm suggests to use the different dynamics of the system to keep the state "turning around the target point (2, 2)" while using few continuous control  $v$ . This can be clearly seen on figure 2.(b) and 2.(c)

- ✓ It can then be argued that the comparison with [32] is unfair since in the latter, the number of switches is deliberately limited by 2. Comparison must be done with the algorithm of [32] in which the same number of switches is allowed. Unfortunately, the optimal solution given by our algorithm uses 65 switches. It is practically impossible to run the algorithms of [13, 12, 32, 27] with such number of switches since this needs somewhere to perform a search over a tree of dimension

$$3^{65}$$

for each choice of the 65 switching instants to explore the set of possible sequences of active configurations.

### 5.3 Example 3

Let us consider the dynamic equations of an induction machine with the norm of flux and the torque as regulated outputs. The voltage input to this machine comes from a two stage power source. The first one is the rectifier that delivers a DC voltage from an AC source and the second is the inverter that transform this DC voltage into a dynamic voltage by changing the positions of the three inverter commutators. This leads to 7 different possible values of the voltage at the input of the induction machine. This is shown on figure 4 in the  $(\alpha, \beta)$ -plane. As a consequence, the system may be viewed as a hybrid system with 7 configurations ( $Q = 7$ ) in each of which, the system is autonomous.

$$\frac{d}{dt} \begin{pmatrix} I_{s\alpha} \\ I_{s\beta} \\ \phi_{s\alpha} \\ \phi_{s\beta} \end{pmatrix} = \begin{pmatrix} -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -p\Omega & \frac{R_r}{z} & \frac{p\Omega L_r}{z} \\ p\Omega & -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -\frac{p\Omega L_r}{z} & \frac{R_r}{z} \\ -R_s & 0 & 0 & 0 \\ 0 & -R_s & 0 & 0 \end{pmatrix} \begin{pmatrix} I_{s\alpha} \\ I_{s\beta} \\ \phi_{s\alpha} \\ \phi_{s\beta} \end{pmatrix} + \begin{pmatrix} \frac{V_0 L_r}{z} & 0 \\ 0 & \frac{V_0 L_r}{z} \\ V_0 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} V_{s\alpha}(q) \\ V_{s\beta}(q) \end{pmatrix} \quad (31)$$

$$y = \begin{pmatrix} \|\Phi\|^2 \\ \Gamma \end{pmatrix} := \begin{pmatrix} \phi_{s\alpha}^2 + \phi_{s\beta}^2 \\ p(I_{s\beta}\phi_{s\alpha} - I_{s\alpha}\phi_{s\beta}) \end{pmatrix} \quad (32)$$

in which,

$$z := L_s L_r - M^2 \quad ; \quad \sigma = 1 - M^2 / (L_s L_r) \quad ; \quad T_s = \frac{L_s}{R_s} \quad ; \quad T_r = \frac{L_r}{R_r}$$

where  $L_s$ ,  $L_r$ ,  $R_s$  and  $R_r$  are the stator and rotor inductances and resistances while  $\Omega$  is the motor speed and  $p = 2$ . The optimal control problem in question here is to find a switching strategy  $q^*(\cdot)$  such that the nonlinear output (32) is regulated around some reference signal  $h_r(t)$  while minimizing the harmonics of the stator current at some precise frequency  $\omega_0$ . To tackle such requirements, the system is extended using additional states as follows

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -p\Omega & \frac{R_r}{z} & \frac{p\Omega L_r}{z} \\ p\Omega & -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -\frac{p\Omega L_r}{z} & \frac{R_r}{z} \\ -R_s & 0 & 0 & 0 \\ 0 & -R_s & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} \frac{V_0 L_r}{z} & 0 \\ 0 & \frac{V_0 L_r}{z} \\ V_0 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} V_{s\alpha}(q) \\ V_{s\beta}(q) \end{pmatrix}$$



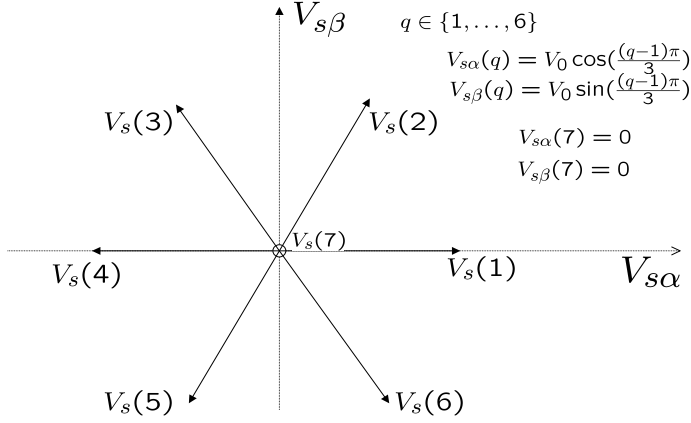


Figure 4: The input voltage of the induction machine as a function of the configuration  $q \in \{1, \dots, 7\}$

(33)

$$\dot{x}_5 = x_1 \cos \omega_0 t \quad ; \quad x_5(0) = 0 \quad (34)$$

$$\dot{x}_6 = x_1 \sin \omega_0 t \quad ; \quad x_6(0) = 0 \quad (35)$$

$$\dot{x}_7 = x_2 \cos \omega_0 t \quad ; \quad x_7(0) = 0 \quad (36)$$

$$\dot{x}_8 = x_2 \sin \omega_0 t \quad ; \quad x_8(0) = 0 \quad (37)$$

$$y = h(x) = \begin{pmatrix} \|\Phi\|^2 \\ \Gamma \end{pmatrix} := \begin{pmatrix} x_3^2 + x_4^2 \\ p(x_2 x_3 - x_1 x_4) \end{pmatrix} \quad (38)$$

Indeed, in doing so, the quantity

$$\Psi(x(T)) := \frac{1}{T^2} \left[ x_5^2(T) + x_6^2(T) + x_7^2(T) + x_8^2(T) \right] \quad (39)$$

is a truncated approximation of the spectral power of the stator current at the prescribed frequency  $\omega_0$  and therefore, the following cost function may be used to reflect the control specifications stated above

$$J(x_0, q) = w_f \Psi(x(T)) + \int_0^T \|h(x(\tau)) - h_r(\tau)\|^2 d\tau \quad (40)$$

where  $w_f$  is a weighting coefficient penalizing the current power spectrum at the frequency  $\omega_0$ . This cost function can be put in the standard form (2) by using the following straightforward definition of  $L_q$  for  $q \in \{1, \dots, 7\}$

$$L_q(x, t) := \frac{\partial \Psi}{\partial x}(x) f_q(x) + \|h(x) - h_r(t)\|^2 \quad (41)$$

where  $f_q(x)$  is clearly defined by the r.h.s of (33)-(37).

The numerical values of the parameters used in the simulations are given on table 2.

The following prediction horizon length  $T$  and sampling period  $h$  are used

$L_s$	$L_r$	$M$	$R_s$	$R_r$	$p$	$\Omega$
0.031747H	0.0323H	0.031H	0.07 $\Omega$	0.052 $\Omega$	2	2 $\pi$ 100rad/sec

Table 2: Numerical values of the induction machine's parameters

$$T = 0.2 \text{ s} \quad ; \quad h = 50 \times 10^{-6} \text{ s}$$

leading to a number of discretization instants  $N = 4000$ . Harmonics attenuation is tested for the value  $\omega_0 = 2\pi \times 170$  rad/s. Finally the reference trajectories on the output are given by the following constant set-points

$$h_r(t) = \begin{pmatrix} 1 \\ 100 \end{pmatrix}$$

The parameters of the algorithm have been taken as follows (these values are used for the normalized system equations)

$$\epsilon_u = \epsilon_J = 0.001 \quad ; \quad d\mu = 1 \quad ; \quad \gamma = 1.2 \quad ; \quad \mu_0 = 0$$

Figure 5 shows the optimal trajectories for both the regulated outputs and the state vector when no penalty is used to insure harmonic rejection ( $w_f = 0$ ). Figure 6 shows the same results when the penalty  $w_f = 10^4$  is used in order to reject the harmonics corresponding to  $w_0 = 2\pi \times 170$  (rad/s). The corresponding power spectrum in the two cases above is given on figure 7 where the effect of the penalty term at the prescribed frequency can be clearly appreciated. Finally, figure 8 shows an example of the optimal switching strategy in the case ( $w_f = 10^4$ ) over the whole time interval (figure 8.(a)) and over a shorter time window enabling the switches to be clearly identified (figure 8.(b)).

## 6 Conclusion and future work

In this paper, an algorithm is proposed to solve optimal control problems for switched hybrid systems. These are hybrid systems with no autonomous switches in which the controlled switches are free. The basic features of the algorithm are the unified framework used in updating both continuous control and switching strategy and the fact that no combinatoric search is needed whatever is the number of switches.

Future works concern the generalization of the proposed scheme to hybrid systems with autonomous switches and/or controlled switched with admissible target configurations that may depend on the active configuration.

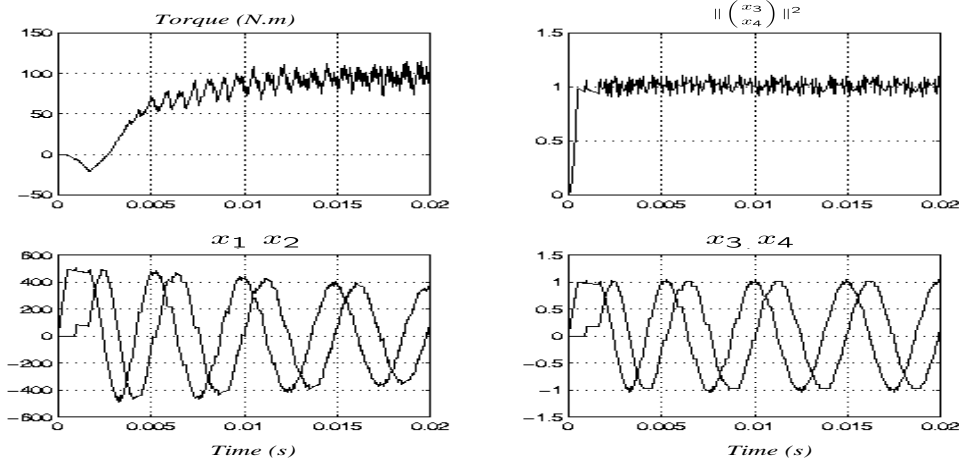


Figure 5: Optimal trajectories without harmonics attenuation ( $w_f = 0$ ).

## A Appendix

First of all, let us state some preliminary results that are extensively used in the following proofs. Using the boundedness condition of assumption 1, the following result can be easily proven using classical arguments concerning the convergence of the Runge-Kutta method.

### PROPOSITION 2 [Convergence of the discretization scheme]

Under assumption 1, for all piece-wise constant admissible control  $\bar{u} \in \bar{U}$ , the solutions  $x(t)$  and  $\lambda(t)$  of (12) and (15) are related to the corresponding approximations  $\bar{x}$  and  $\bar{\lambda}$  given by (20) and (21) by the following inequalities

$$\|x(t_k) - \bar{x}(k)\| \leq \rho_1(h) \quad ; \quad \text{for all } k = 1 \dots N \quad (42)$$

$$\|\lambda(t_k) - \bar{\lambda}(k)\| \leq \rho_2(h) \quad ; \quad \text{for all } k = 1 \dots N \quad (43)$$

Furthermore, the approximate cost estimation error satisfies

$$|\bar{J}(\bar{u}) - J(\bar{u})| \leq \rho_3(h) \quad (44)$$

where the  $\rho_i(\cdot)$ 's are some functions depending on the admissible set  $U$  and such that  $\lim_{h \rightarrow 0} |\rho_j(h)| = 0$  ( $j \in \{1, 2, 3\}$ ). ♠

By manipulating (15) using the boundedness assumption together with the well known Gronwall inequality

### PROPOSITION 3 [Boundness of the adjoint state evolution]

Under assumption 1, there is a positive constant  $M_2 > 0$  such that for all admissible control  $u$ , the solution  $\lambda(\cdot)$  of (15) in which  $x(\cdot)$  stands for the solution of (12) under the control strategy  $u(\cdot)$  satisfies the following inequality

$$\|\lambda(t)\| \leq M_2 \quad \text{for all } t \in [t_0, t_1] \quad (45)$$

♠

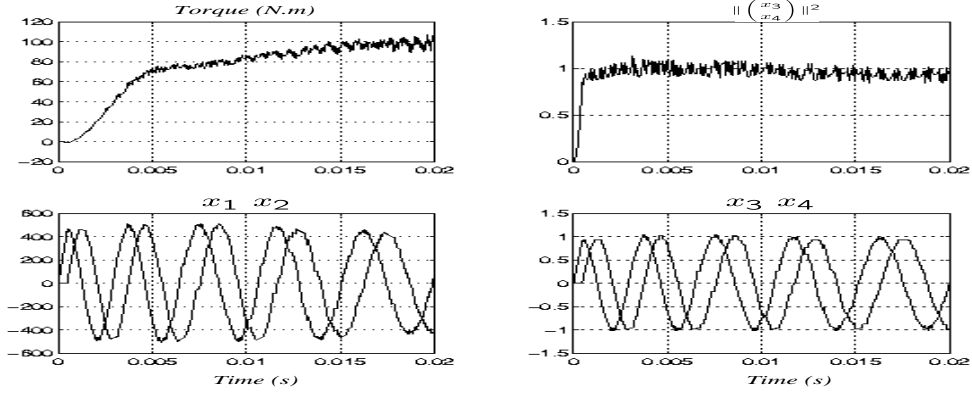


Figure 6: Optimal trajectories without harmonics attenuation ( $w_f = 10^4$ ).

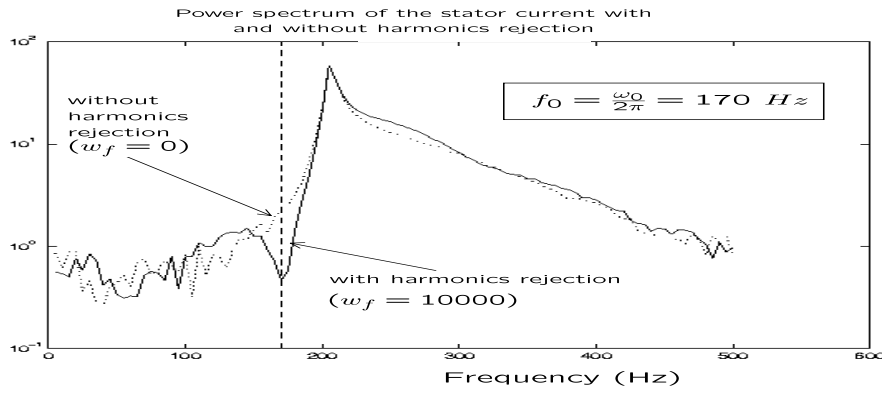


Figure 7: Power spectrum of the stator current for  $w_f = 0$  (dotted lines) and  $w_f = 10^4$  (solid lines).

An immediate consequence of the above proposition is the following

**COROLLARY 3 [Boundedness of the generated sequences]**

Under assumption 1, for all  $\epsilon > 0$ , there is a sufficiently small  $h > 0$  such that the discretization scheme defined by (18), (20) and (21) using any admissible piece-wise constant control  $\bar{u} \in \bar{U}$  leads to solutions  $\bar{x}, \bar{\lambda}$  such that

$$\| \bar{x}(k) \| \leq M_1 + \epsilon \quad ; \quad \forall k = 1 \dots N \tag{46}$$

$$\| \bar{\lambda}(k) \| \leq M_2 + \epsilon \quad ; \quad \forall k = 1 \dots N \tag{47}$$



**Proof**

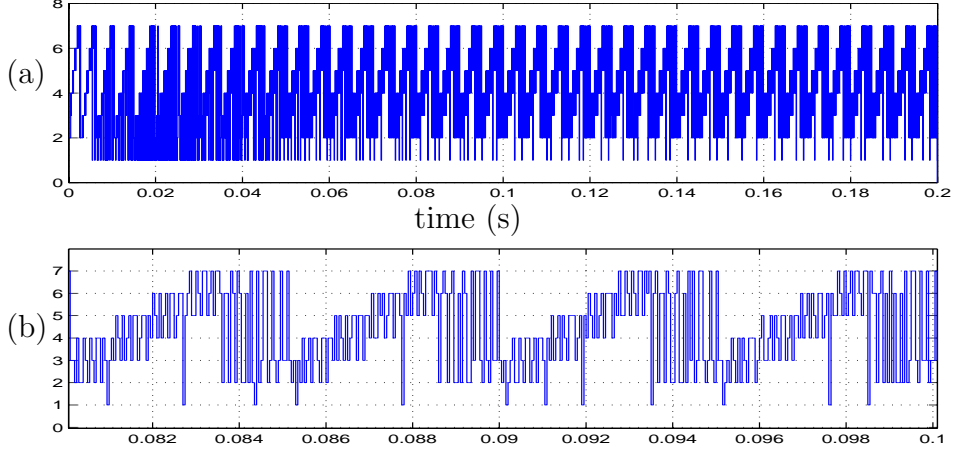


Figure 8: The optimal switching strategy for the case  $w_f = 10^4$ .

A straightforward consequence of (42)-(43) and (14)-(45) when  $h > 0$  is sufficiently small for the following inequality to hold

$$\max(\rho_1(h), \rho_2(h)) \leq \epsilon \quad (48)$$

◇

In what follows, we suppose that the step length  $h$  has been chosen once for all and such that corollary 3 holds for  $\epsilon = 1$  (for instance). Therefore, all the sequences  $(\bar{x}^i, \bar{u}^i, \bar{\lambda}^i, t_k)$  generated by the above algorithm lie in the compact set

$$\mathcal{S} := \bar{X} \times \bar{U} \times \bar{\Lambda} \times [t_0, t_f] \subset \mathfrak{R}^{2Nn+(N-1)m+1} \quad (49)$$

where,

$$\bar{X} := \bar{B}(0, M_1 + 1) \times \dots \times \bar{B}(0, M_1 + 1) \subset \mathfrak{R}^{Nn} \quad (50)$$

$$\bar{\Lambda} := \bar{B}(0, M_2 + 1) \times \dots \times \bar{B}(0, M_2 + 1) \subset \mathfrak{R}^{Nn} \quad (51)$$

## A.1 Proof of proposition 1

We have by definition (22)

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) = h \sum_{k=1}^{N-1} \left[ L(\bar{x}^i(k), \bar{u}^i(k), t_k) - L(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), t_k) \right] \quad (52)$$

using the definition (16) of the Hamiltonian, we can write

$$\begin{aligned} L(\bar{x}^i(k), \bar{u}^i(k), t_k) &= H(\bar{x}^i(k), \bar{u}^i(k), \bar{\lambda}^{i-1}(k), t_k) - f^T(\bar{x}^i(k), \bar{u}^i(k), t_k) \bar{\lambda}^{i-1}(k) \\ L(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), t_k) &= H(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) - f^T(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), t_k) \bar{\lambda}^{i-1}(k) \end{aligned}$$

injecting the last two equations into (52) gives

$$\begin{aligned}\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) &= h \sum_{k=1}^{N-1} \left[ H(\bar{x}^i(k), \bar{u}^i(k), \bar{\lambda}^{i-1}(k), t_k) - H(\bar{x}^i(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \right. \\ &+ H(\bar{x}^i(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) - H(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \\ &\left. - \left( f^T(\bar{x}^i(k), \bar{u}^i(k), t_k) - f^T(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), t_k) \right) \bar{\lambda}^{i-1}(k) \right] \quad (53)\end{aligned}$$

we shall successively examine each group of two consecutive terms in (53).

**For the first one** we have by definition of  $\bar{u}^i$  (see step 3 of the algorithm)

$$H(\bar{x}^i(k), \bar{u}^i(k), \bar{\lambda}^{i-1}(k), t_k) - H(\bar{x}^i(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \leq -\mu^{i-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 \quad (54)$$

as for **the second group**, we can write (using  $\delta\bar{x}^i$  to denote  $\bar{x}^i - \bar{x}^{i-1}$ )

$$\begin{aligned}H(\bar{x}^i(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) - H(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) &= \\ H_x(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \delta\bar{x}^i(k) + (\delta\bar{x}^i(k))^T H_{xx}(\theta^i, \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \delta\bar{x}^i(k) \quad (55)\end{aligned}$$

where  $\theta^i \in \bar{X}$ . Now if we define  $r_1 > 0$  by

$$r_1 = \sup_{z \in \mathcal{S}} \|H_{xx}(z)\| \quad (56)$$

then (55) gives

$$\begin{aligned}H(\bar{x}^i(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) - H(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) &\leq \\ H_x(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \delta\bar{x}^i(k) + r_1 \|\delta\bar{x}^i(k)\|^2 \quad (57)\end{aligned}$$

using (54) and (57) into (53), we obtain

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h \left[ -\mu^{i-1} \sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 + r_1 \sum_{k=1}^{N-1} \|\delta\bar{x}^i(k)\|^2 \right] + I^i(h) \quad (58)$$

where

$$\begin{aligned}I^i(h) &:= h \sum_{k=1}^{N-1} \left[ H_x(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) \delta\bar{x}^i(k) \right. \\ &\left. - \left( f^T(\bar{x}^i(k), \bar{u}^i(k), t_k) - f^T(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), t_k) \right) \bar{\lambda}^{i-1}(k) \right] \quad (59)\end{aligned}$$

However, the convergence of the discretization scheme implies that

$$H_x^T(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), \bar{\lambda}^{i-1}(k), t_k) = -\frac{\bar{\lambda}^{i-1}(k+1) - \bar{\lambda}^{i-1}(k)}{h} + O(h) \quad (60)$$

$$\left( f^T(\bar{x}^i(k), \bar{u}^i(k), t_k) - f^T(\bar{x}^{i-1}(k), \bar{u}^{i-1}(k), t_k) \right) = \frac{\delta\bar{x}^i(k+1) - \delta\bar{x}^i(k)}{h} + O(h) \quad (61)$$

Let us use the following notations

$$\Delta(\delta\bar{x}^i(k)) := \delta\bar{x}^i(k+1) - \delta\bar{x}^i(k) \quad ; \quad k = 1 \dots N-1 \quad (62)$$

$$\Delta(\bar{\lambda}^i(k)) := \bar{\lambda}^i(k+1) - \bar{\lambda}^i(k) \quad ; \quad k = 1 \dots N-1 \quad (63)$$

rewriting of (59) using (60)-(63) gives

$$I^i(h) = - \sum_{k=1}^{N-1} \left[ \Delta(\bar{\lambda}^{i-1}(k))^T \delta\bar{x}^i(k) + \Delta(\delta\bar{x}^i(k))^T \bar{\lambda}^{i-1}(k) + o(h) \right] \quad (64)$$

$$= - \sum_{k=1}^{N-1} \left[ \Delta(\delta^T \bar{x}^i(k) \bar{\lambda}^{i-1}(k)) - \underbrace{\Delta^T(\bar{\lambda}^{i-1}(k)) \Delta(\bar{x}^i(k))}_{=O(h).O(h)=o(h)} + o(h) \right] \quad (65)$$

$$= - \sum_{k=1}^{N-1} \left[ \Delta(\delta^T \bar{x}^i(k) \bar{\lambda}^{i-1}(k)) + o(h) \right] \quad (66)$$

recall that  $h = \frac{t_f - t_0}{N-1}$ , hence

$$\sum_{k=1}^{N-1} o(h) = O(h) \quad (67)$$

and (66) becomes

$$I^i(h) = - \sum_{k=1}^{N-1} \left[ \Delta(\delta^T \bar{x}^i(k) \bar{\lambda}^{i-1}(k)) \right] + O(h) \quad (68)$$

$$= \delta^T \bar{x}^i(1) \bar{\lambda}^{i-1}(1) - \delta^T \bar{x}^i(N) \bar{\lambda}^{i-1}(N) + O(h) \quad (69)$$

however, according to the boundary conditions on  $x$  and  $\lambda$  (respected by the algorithm), we have  $\bar{\lambda}^{i-1}(N) = 0$  and  $\delta\bar{x}^i(1) = 0$  for all  $i$ , therefore

$$I^i(h) = O(h) \quad (70)$$

this with (58) gives

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h \left[ -\mu^{i-1} \sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 + r_1 \sum_{k=1}^{N-1} \|\delta\bar{x}^i(k)\|^2 \right] + O(h) \quad (71)$$

We shall prove hereafter that there is positive constant  $r_2 > 0$  such that

$$\sum_{k=1}^{N-1} \|\delta\bar{x}^i(k)\|^2 \leq r_2 \cdot \sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 \quad (72)$$

for this, let us write (20) in the following compact form

$$\bar{x}(k+1) := \bar{x}(k) + hF^{(k)}(\bar{x}(k), \bar{u}(k), t_k, t_{k+1}) \quad (73)$$

where  $F^{(k)}$  is clearly defined by identification with (20). Note also that according to the regularity assumptions,  $F^{(k)}$  is  $C^2$  in its arguments.

From (73) we can write

$$\delta\bar{x}^i(k+1) = h \left[ F_x^{(k)}(\theta^i(k), w^i(k), t_k, t_{k+1}) \delta\bar{x}^i(k) + F_u^{(k)}(\theta^i(k), w^i(k), t_k, t_{k+1}) [\bar{u}^i(k) - \bar{u}^{i-1}(k)] \right] \quad (74)$$

where  $\theta^i(k) \in \bar{X}$  and  $w^i(k)$  belongs to the convex hull  $[\bar{U}]$  of  $\bar{U}$ . Therefore, using  $\gamma_1$  and  $\gamma_2$  defined by

$$\gamma_1 = \sup_{(\theta, w, \tau_1, \tau_2) \in \bar{X} \times [\bar{U}] \times [t_0, t_f]^2} \max_{k=1 \dots N-1} \| F_x^{(k)}(\theta, w, \tau_1, \tau_2) \| \quad (75)$$

$$\gamma_2 = \sup_{(\theta, w, \tau_1, \tau_2) \in \bar{X} \times [\bar{U}] \times [t_0, t_f]^2} \max_{k=1 \dots N-1} \| F_u^{(k)}(\theta, w, \tau_1, \tau_2) \| \quad (76)$$

equation (74) can be rewritten as follows

$$\| \delta\bar{x}^i(k+1) \| \leq h \left[ \gamma_1 \| \delta\bar{x}^i(k) \| + \gamma_2 \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \| \right] \quad ; \quad \delta\bar{x}^i(1) = 0 \quad (77)$$

which gives after some manipulations

$$\begin{aligned} \| \delta\bar{x}^i(k) \| &\leq h\gamma_2 \sum_{j=2}^k \gamma_1^{j-2} \| \bar{u}^i(j-1) - \bar{u}^{i-1}(j-1) \| \quad ; \quad k \geq 2 \\ &\leq h\gamma_2 \sum_{j=1}^{k-1} \gamma_1^{j-1} \| \bar{u}^i(j) - \bar{u}^{i-1}(j) \| \quad ; \quad k \geq 2 \end{aligned} \quad (78)$$

now using  $\bar{\gamma}_1 := \max_{j=1 \dots N-1} \{ \gamma_1^j \}$ , we obtain

$$\| \delta\bar{x}^i(k) \| \leq h\gamma_2 \bar{\gamma}_1 \sum_{j=1}^{N-1} \| \bar{u}^i(j) - \bar{u}^{i-1}(j) \| \quad ; \quad k \geq 2 \quad (79)$$

therefore, using Cauchy-Schawrtz inequality and summing (79) for  $k=1$  to  $k=N-1$

$$\sum_{k=1}^{N-1} \| \delta\bar{x}^i(k) \|^2 \leq h^2 (\gamma_2 \bar{\gamma}_1)^2 (N-1) \sum_{j=1}^{N-1} \| \bar{u}^i(j) - \bar{u}^{i-1}(j) \|^2 = \frac{(t_f - t_0)^2}{N-1} (\gamma_2 \bar{\gamma}_1)^2 \sum_{k=1}^{N-1} \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \|^2 \quad (80)$$

which is nothing else than (72) with  $r_2 = \frac{(t_f - t_0)^2}{N-1} (\gamma_2 \bar{\gamma}_1)^2$ .

Hence, (71) can be written in the form (taking  $r := r_1 r_2$ )

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h(r - \mu^{i-1}) \sum_{k=1}^{N-1} \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \|^2 + O(h) \quad (81)$$

which gives (23) whenever  $h$  is sufficiently small. This ends the proof of proposition 1  $\diamond$



## A.2 Proof of Corollary 1

Let us consider the following norm

$$\| \bar{u}^i - \bar{u}^{i-1} \|_\infty := \max_{k=1 \dots N-1} \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \| \quad (82)$$

When we consider the sequence  $(\bar{u}^i)$ , for each  $i$ , two situations are possible

- $\| \bar{u}^i - \bar{u}^{i-1} \|_\infty = 0$ , in this case,  $\bar{J}(\bar{u}^i) = \bar{J}(\bar{u}^{i-1})$ ,
- $\| \bar{u}^i - \bar{u}^{i-1} \|_\infty > 0$ , in this case, according to (23) of proposition 1, it is always possible to find sufficiently high  $\mu^{i-1}$  (obtained after successive application of  $\mu^{i-1} = \mu^{i-1} + d\mu$  of **step 4**) such that the condition

$$\bar{J}(\bar{u}^i) \leq \bar{J}(\bar{u}^{i-1}) - \epsilon_J \quad (83)$$

necessary to leave step **step 4** is satisfied. Therefore, in this case, we have  $\bar{J}(\bar{u}^i) < \bar{J}(\bar{u}^{i-1})$

Therefore,  $\bar{J}(\bar{u}^i)$  can only decrease. Now, when  $i_{max} = \infty$ , the algorithm generates an infinite decreasing sequence  $\{\bar{J}(\bar{u}^i)\}$  that it is bounded below hence convergent.  $\diamond$

## A.3 Proof of Corollary 2

In order to prove 1), it is sufficient to prove that there is only a finite number of integer  $i$  such that

$$\| \bar{u}^i - \bar{u}^{i-1} \|_\infty > 0$$

where  $\| \cdot \|_\infty$  is defined by (82). Suppose that this is not true, then there will be an infinite subsequence  $i_j$  of indices such that  $\| \bar{u}^{i_j} - \bar{u}^{i_j-1} \|_\infty > 0$ . According to the proof of corollary 1, however, this subsequence must satisfy

$$\bar{J}(\bar{u}^{i_j}) \leq \bar{J}(\bar{u}^{i_j-1}) - \epsilon_J$$

and  $\bar{J}(\bar{u}^{i_j})$  tends to  $-\infty$  when  $j$  tends to  $\infty$ . This is impossible because the cost function values are bounded below.

By definition of  $\bar{u}^i$  (see **Step 3**), we can write for all  $k = 1 \dots N - 1$

$$H(\bar{x}^i(k), \bar{u}^i, \bar{\lambda}^{i-1}, t_k) + \mu^{i-1} \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \|^2 \leq H(\bar{x}^i(k), \bar{u}, \bar{\lambda}^{i-1}, t_k) + \mu^{i-1} \| \bar{u}(k) - \bar{u}^{i-1}(k) \|^2 \quad \forall \bar{u} \in \bar{U} \quad (84)$$

Now taking the subsequence  $i_j$  such that  $\lim_{j \rightarrow \infty} \mu^{i_j-1} = 0$ , and taking the limit gives

$$H(\bar{x}^*(k), \bar{u}^*, \bar{\lambda}^*, t_k) \leq H(\bar{x}^*(k), \bar{u}, \bar{\lambda}^*, t_k) \quad \forall \bar{u} \in \bar{U}. \quad (85)$$

This ends the proof of 1).

To prove 2) it is sufficient to remark that if  $\bar{u}^{\bar{i}} = \bar{u}^{\bar{i}-1} =: \bar{u}^*$  with  $\mu^{\bar{i}-1} = 0$  then the sequence  $\bar{u}^i$  for  $i \geq \bar{i}$  becomes constant and  $\bar{u}^*$  is a trivial accumulation point of the control sequence. The result follows from point 1).

To prove 3), it is sufficient to note that, using the result of 1), the algorithm stops when  $i$  satisfies

$$i \geq \max\{i_{max}, \bar{i}\}$$

(see the stop condition of **Step 6**). ♠

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