Hybrid System Theory Based Optimal Control of an Electronic Throttle

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FER

Overview

- PWA Model of an Electronic Throttle
- Finite Time Optimal Control
- Multiparametric Mixed Integer Programming
- mp-MIQP Algorithm
- Results

Electronic Throttle

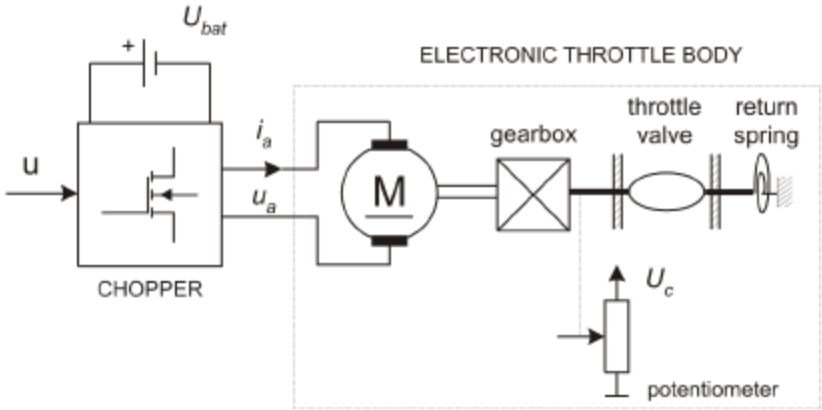
Advantages

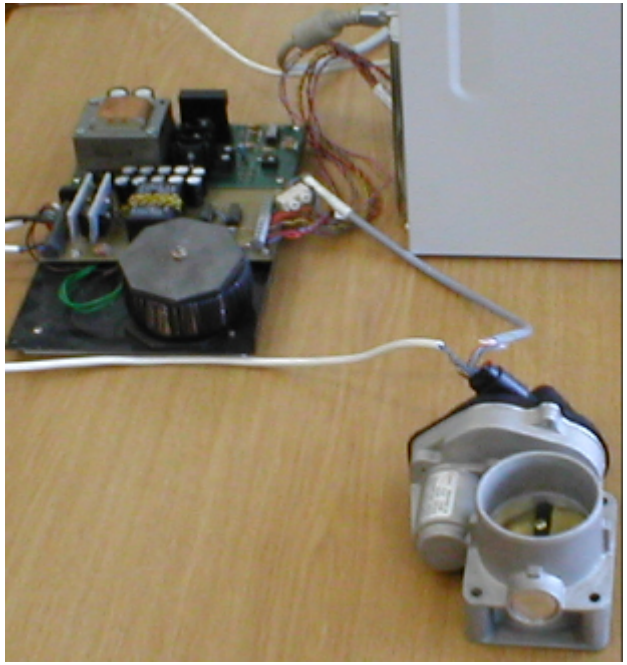
- Improved fuel economy
- Reduction in vehicle emission
- Improved drivability

Components

- DC drive
- Gearbox
- Return spring
- Dual potentiometer

Control Challenges

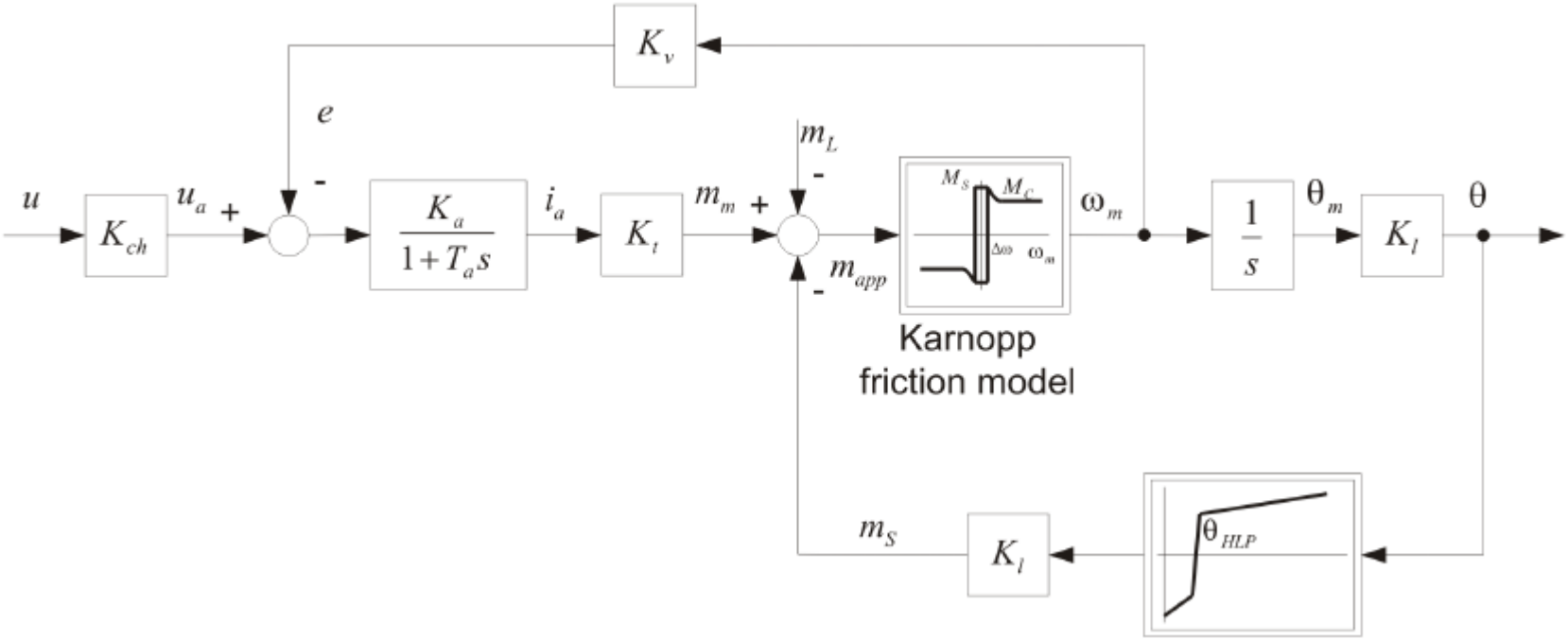




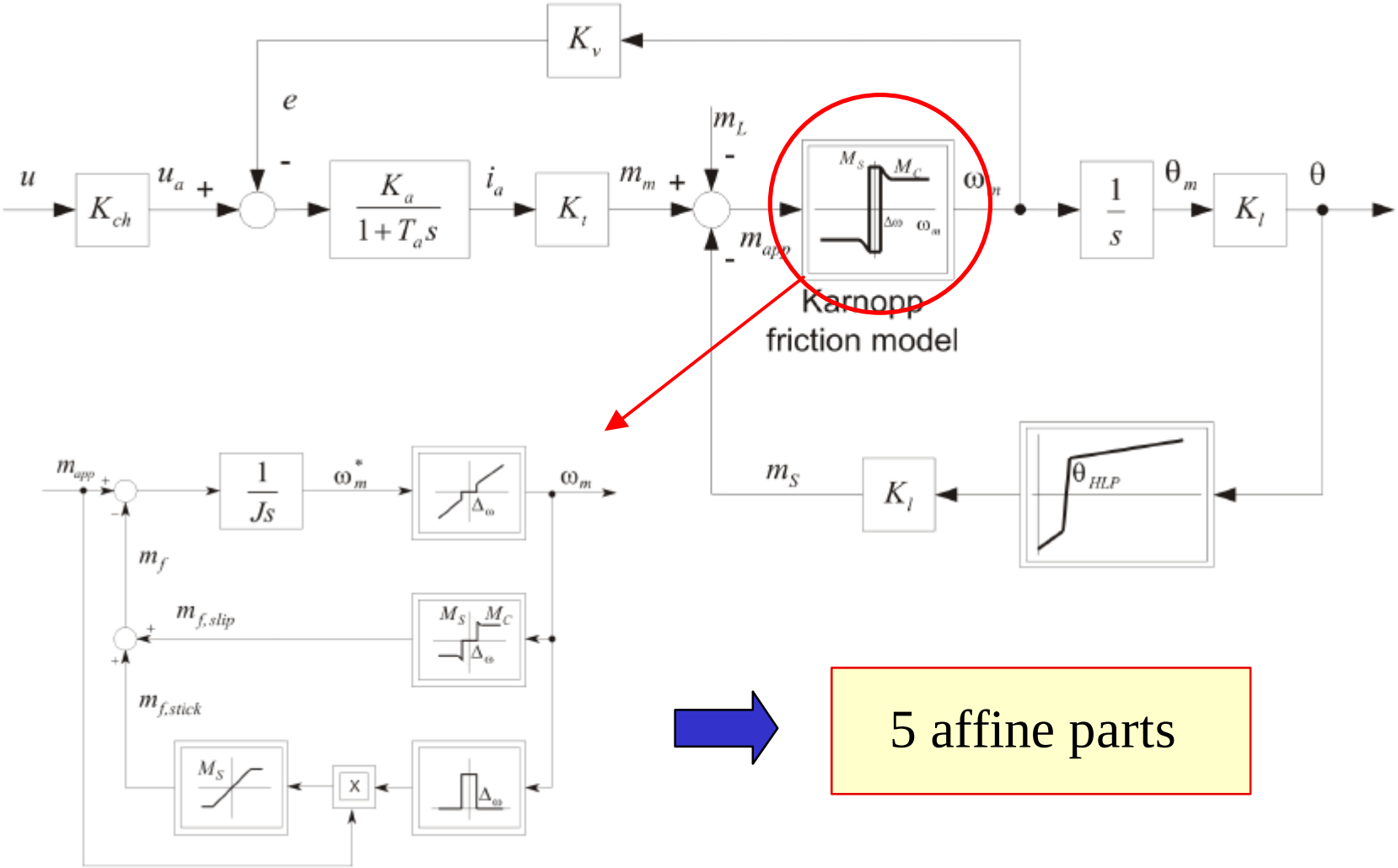
Main challenges

- Friction (gearbox)
- Limp-Home nonlinearity (return spring)
- Quantization (dual potentiometer + A/D)
- Constraints

Nonlinear Model of the Throttle

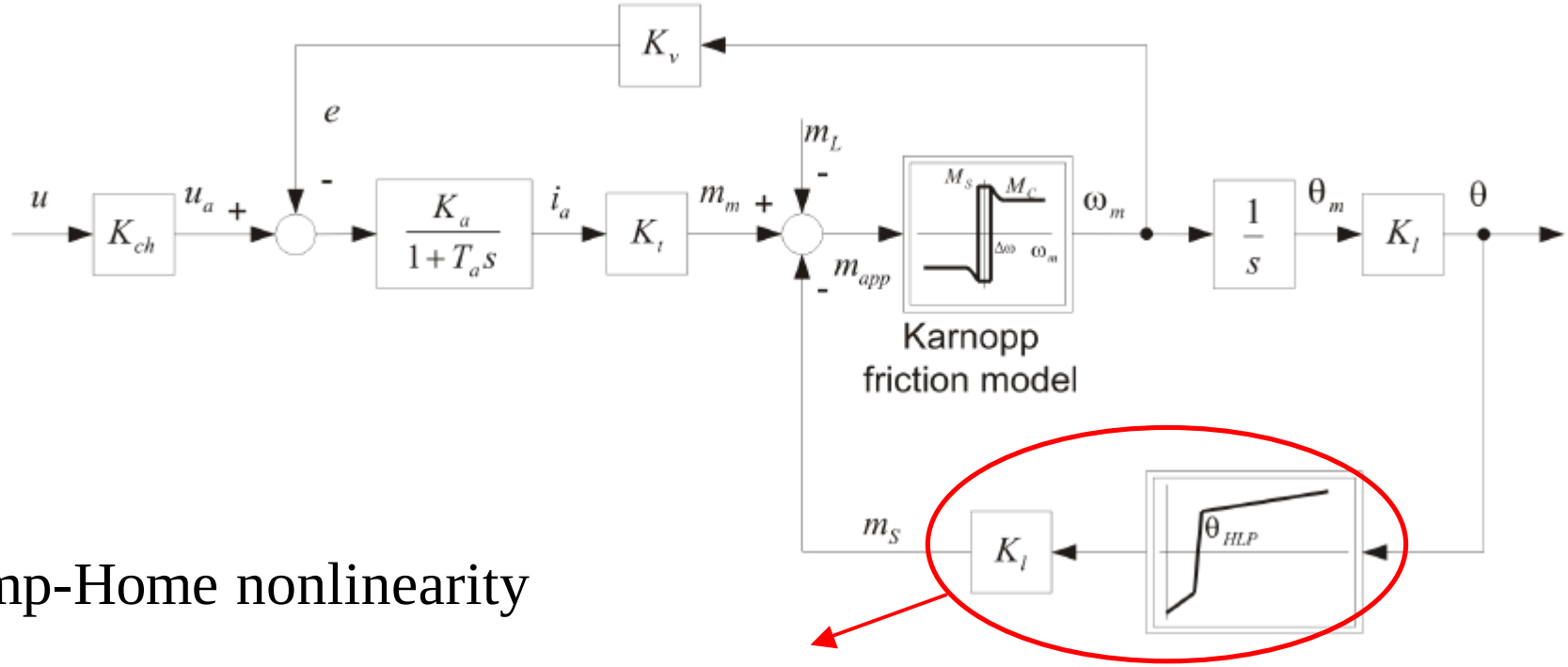


Nonlinear Model of the Throttle



5 affine parts

Nonlinear Model of the Throttle



Limp-Home nonlinearity

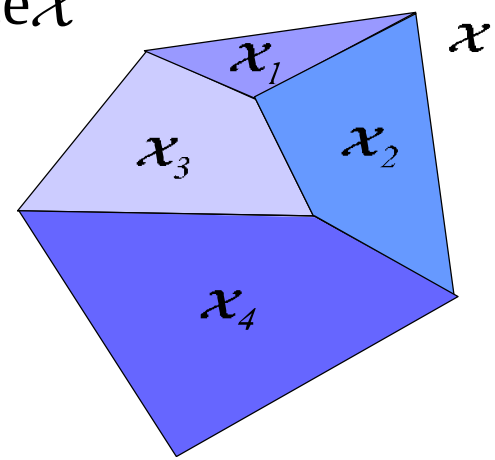
$$m_S = \begin{cases} K_l(K_{S11}q + K_{S01}) & \text{if} \quad q \le q_{LHL} \\ K_l(K_{S12}q + K_{S02}) & \text{if} \quad q_{LHL} \le q \le q_{LHH} \\ K_l(K_{S13}q + K_{S03}) & \text{if} \quad q \ge q_{LHH} \end{cases}$$

3 affine parts

Piecewise Affine System

$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i \\ y(t) &= C_i x(t) + g_i \\ i &= 1, \ldots, s \end{aligned} \qquad \text{if} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i$$

- $\{\mathcal{X}_i\}_1^s$ polyhedral partition of *state* and *input* space $\mathcal{X}$
- $x \in \mathrm{R}^{n_c} \times \{0,1\}^{n_\ell}, \quad n \triangleq n_c + n_\ell$
- $u \in \mathrm{R}^{m_c} \times \{0,1\}^{m_\ell}, \quad m \triangleq m_c + m_\ell.$
- switches both internal and controllable
- state and input constraints are included in $\mathcal{X}$



PWA systems include many hybrid system classes

PWA model of the Throttle

- 5 affine parts from the friction
- 3 affine parts from the LH
- sampling with Zero Order Hold

$\Rightarrow$ 15 (discrete time) PWA dynamics

With state vector $x = [w_m^*, q]^T$ we get PWA system

$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i \\ y(t) &= C_i x(t) + g_i \\ i &= 1, \ldots, 15 \end{aligned} \qquad \text{if} \qquad H_i x(t) + L_i u(t) \leq K_i$$

where polyhedron description $H_i x(t) + L_i u(t) \leq K_i(t)$

also includes state and input constraints

Finite Time Optimal Control

- Performance Index

$$J^*(x) = \min_U \|x_N\| + \sum_{k=0}^{N-1} \|x_k\| + \|u_k\|$$

- Constraints

$$\text{subject to} \begin{cases} y_{\min} \le y_k \le y_{\max}, & k = 1,\ldots,N \\ u_{\min} \le u_k \le u_{\max}, & k = 0,\ldots,N-1 \\ x_{k+1} = Ax_k + Bu_k, & k \ge 0 \end{cases}$$

$$\left.\begin{array}{lll} \text{For } \|\bullet\|_{1,\infty} & \Rightarrow & \text{LP} \\ \text{For } \|\bullet\|_2 & \Rightarrow & \text{QP} \end{array}\right\} \Rightarrow \text{Optimizer: } U^*(x) = \{u_0^*, u_1^*, \ldots, u_{N-1}^*\}$$

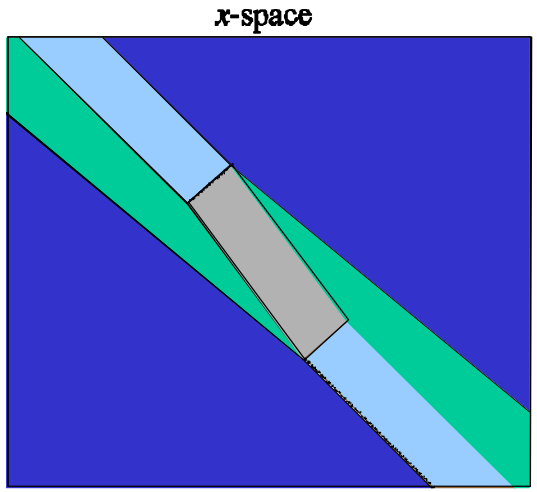
Receding Horizon Policy: $u = u_0^* = f(x)$

Receding Horizon Policy $u = f(x)$

- Solve LP/QP **for each** x
- Solve LP/QP **for all** x via multiparametric program

Piecewise affine state-feedback law

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \le K_1 \\ \vdots & \qquad\quad \vdots \\ F_{N_r} x + G_{N_r} & \text{if} \quad H_{N_r} x \le K_{N_r} \end{cases}$$



Solution obtained via:

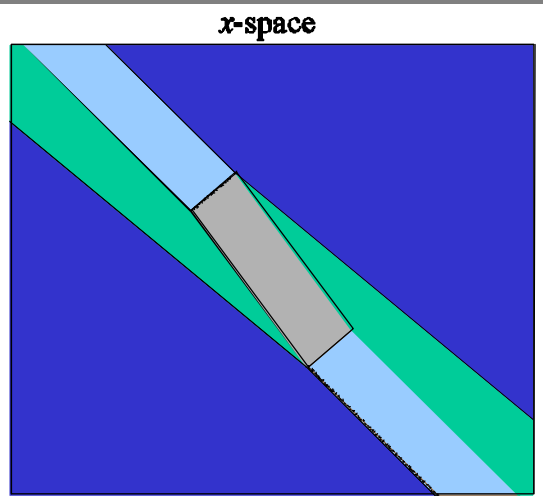
mp-LP for $\|\bullet\|_{1,\infty}$ (Bemporad, Borrelli, Morari, CDC 2000)

mp-QP for $\|\bullet\|_{2}$ (Bemporad, Morari, Dua, Pistikopoulos, Automatica 2001)

mp-QP Properties

- Solution $u^*(x)$ **is** continuous, piecewise affine
  - the set X* is partitioned into polyhedral critical regions
  - in each critical region $u^*(x)$ is an affine function of x
- Value function $J^*(x)$ is convex, piecewise quadratic, and continuously differentiable if the mp-QP is not degenerate

$$u^*(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \le K_1 \\ \vdots & \qquad\quad \vdots \\ F_{N_r} x + G_{N_r} & \text{if} \quad H_{N_r} x \le K_{N_r} \end{cases}$$



Optimal Control of the PWA system

$$
\begin{array}{ll}
\min_U & x(N)'Px(N) + \sum_{k=0}^{N-1} x(k)'Qx(k) + u(k)'Ru(k) \\
\text{subj.to} & \boxed{\begin{array}{l} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \qquad\qquad \text{if } [x(t), u(t)] \in \mathcal{X}^i, \quad i = 1, \ldots, s \end{array}} \\
& Ex(k) + Lu(k) \leq M, \quad k = 0, \ldots, N-1 \\
& x(N) \in \mathcal{X}^f
\end{array}
$$

- $U = \{u(0), \ldots, u(N-1)\}$ is the optimizer
- Extendable to penalize switches,…

Feedback solution?

How do we compute it?

Translation into Mixed Integer Program

$$\min_{U^N} J(U^N, x(0)) \triangleq \|Px(N)\|_p + \sum_{k=0}^{N-1} \|Qx(k)\|_p + \|Ru(k)\|_p$$

$$\text{subj. to} \begin{cases} x(N) \in \mathcal{X}^f \\ x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \\ y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) \\ E_2\delta(k) + E_3z(k) \leq E_1u(k) + E_4x(k) + E_5 \end{cases}$$

Equivalent MLD representation of the PWA system

(Bemporad, Morari 1999)

Mixed Integer Program

$$\min_{\epsilon} \ \epsilon^T H \epsilon + (f^T + x(0)^T F)\epsilon$$

$$\text{s.t.} \ \ G\epsilon \leq w + Fx(0)$$

Multiparametric Mixed Integer Program

$$\min_{\epsilon} \quad \epsilon^T H \epsilon + (f^T + x(0)^T F)\epsilon$$
$$\text{s.t.} \quad G\epsilon \leq w + Fx(0)$$

is solved for all $x(0)$ by using

- p = 1/∞ ⇒ Multiparametric mixed integer linear programming (H=0)
- p = 2 ⇒ Multiparametric mixed integer quadratic programming ($H \neq 0$)

to compute

$$u^*(x(0)) = f_{PWA}(x(0))$$

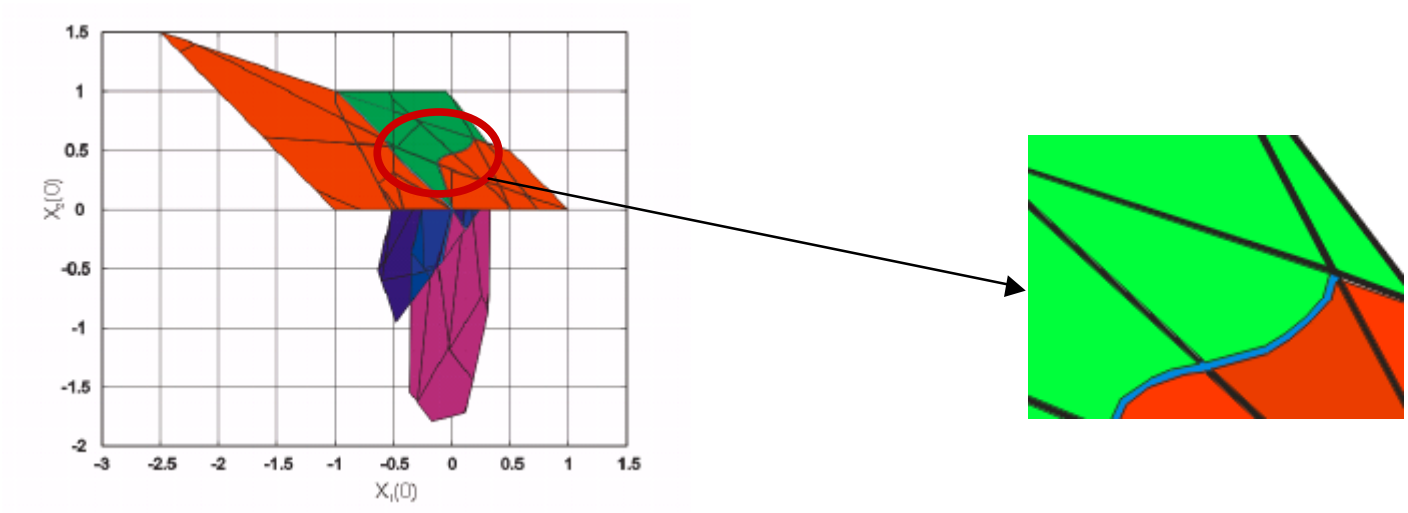
Characterization of the Solution (p=2)

(Sontag 1981, Mayne 2001)

The solution to the optimal control problem is a time varying PWA state feedback control law of the form

$$u^*(x(k)) = F_i^k x(k) + G_i^k$$
$$\text{if} \quad x(k) \in \mathcal{P}_i^k \triangleq \{x : x' L_i^k(j) x + M_i^k(j) x \leq K_i^k(j)\}$$

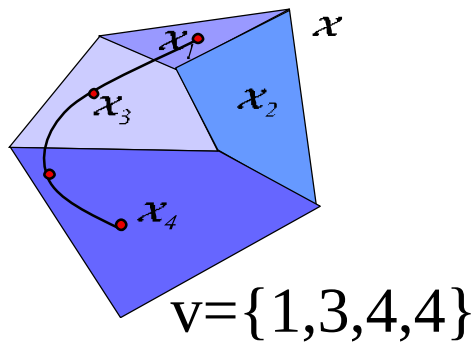
$\{\mathcal{P}_i^k\}_{i=1}^{N_k}$ partition of the set $\mathcal{X}_k^*$ of feasible states $x(k)$.



Characterization of the Solution: Proof

$$\begin{aligned} x(t+1) &= A_i x(t) + B_i u(t) + f_i \\ y(t) &= C_i x(t) + g_i \end{aligned} \qquad \text{for} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i$$

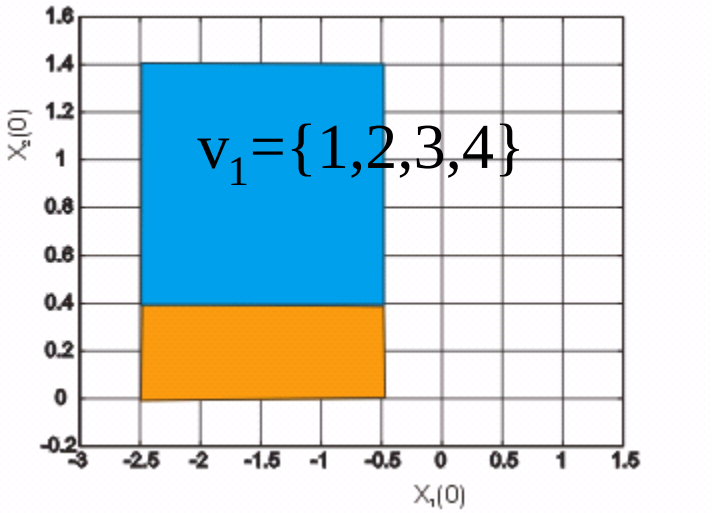
- Fix a switching sequence of length N to obtain *constrained linear time variant system*

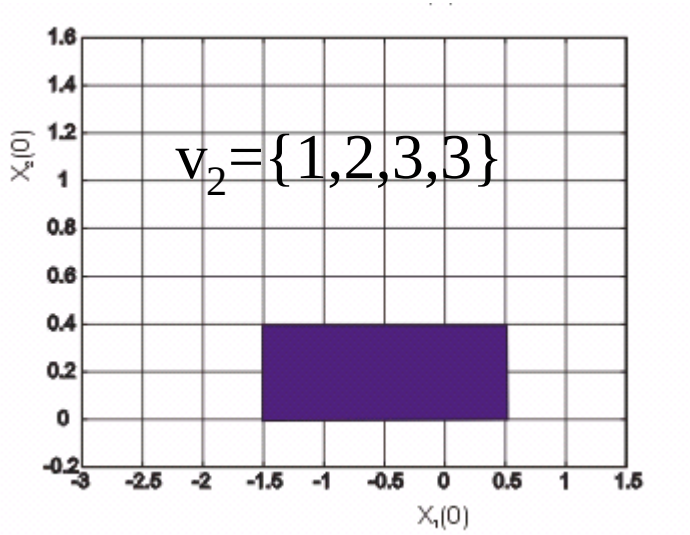


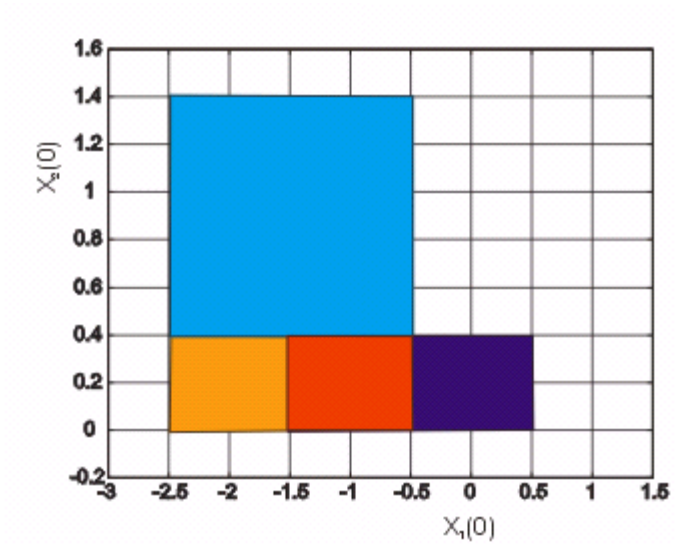
- Solve finite time optimal control to obtain state-space *polyhedral* partition and corresponding PWA input and PWQ value function

Obtain optimal control law by comparing value functions on polyhedron of "multiple feasibility"

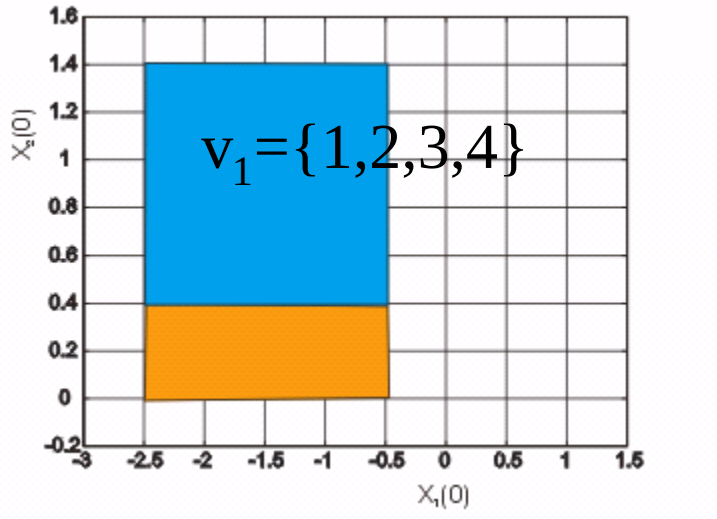
Polyhedron of "multiple feasibility"

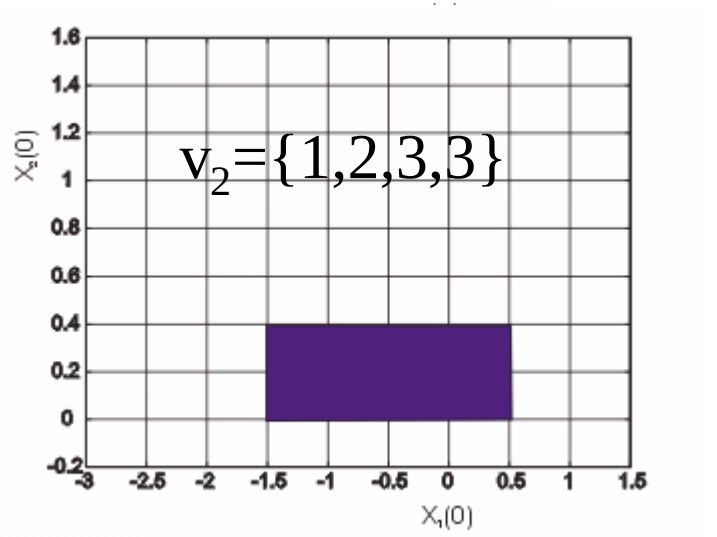


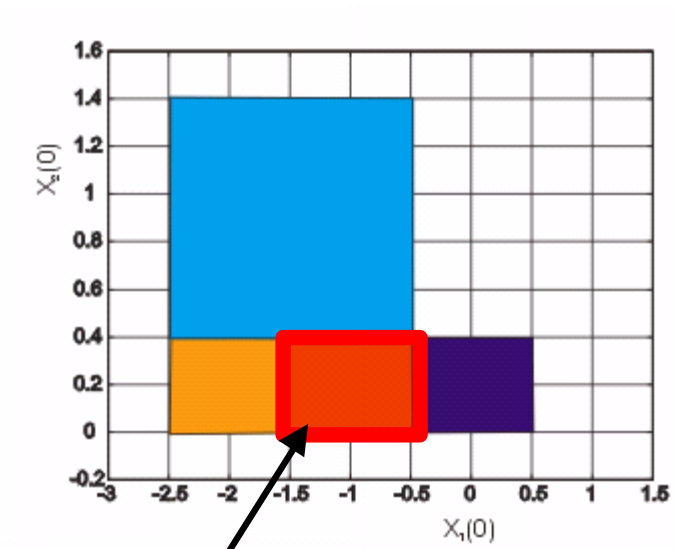




Polyhedron of "multiple feasibility"

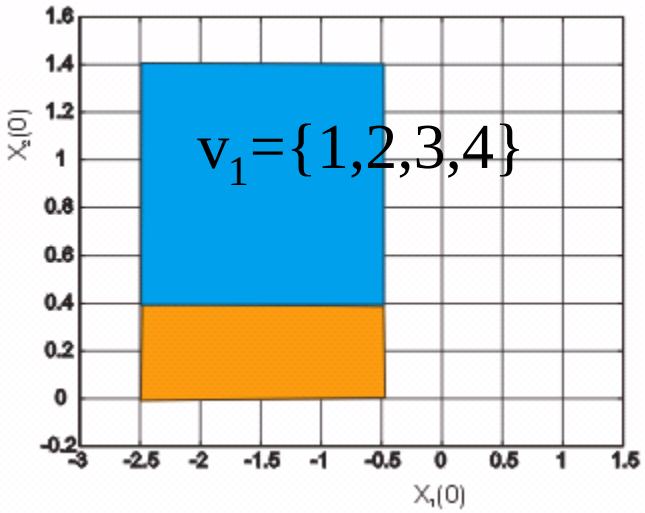


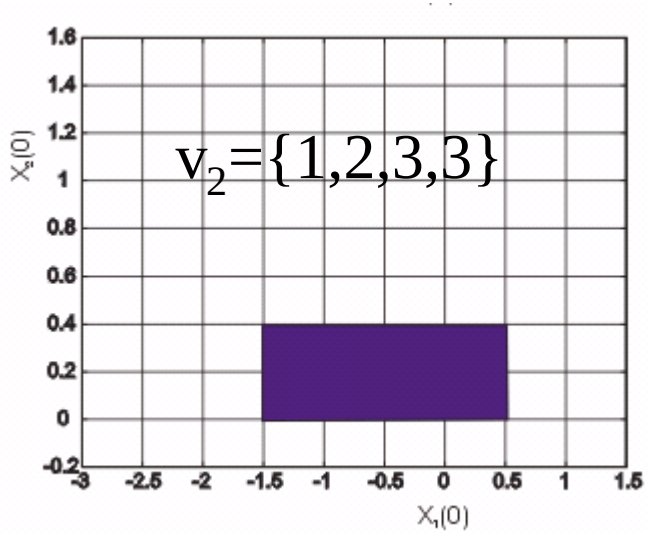


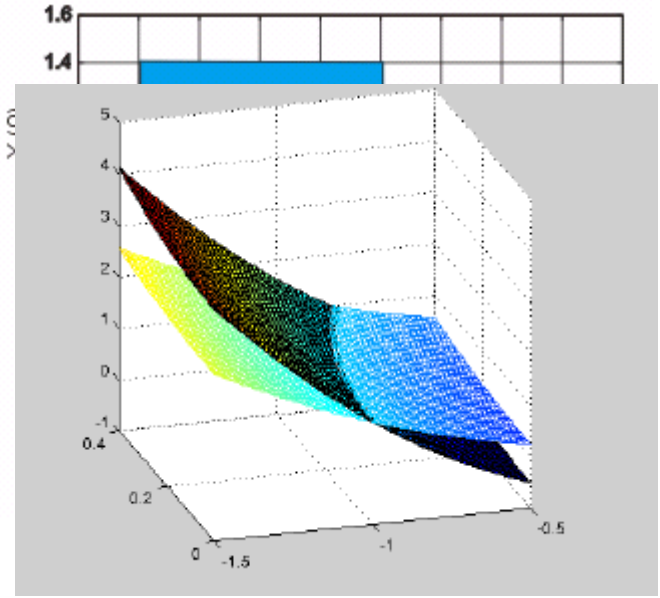


Polyhedron of multiple feasibility: switch v_1 and v_2 both admissible

Polyhedron of "multiple feasibility"







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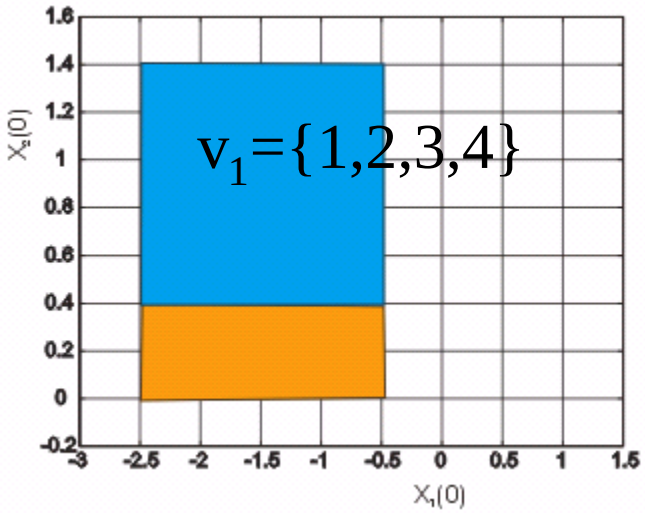
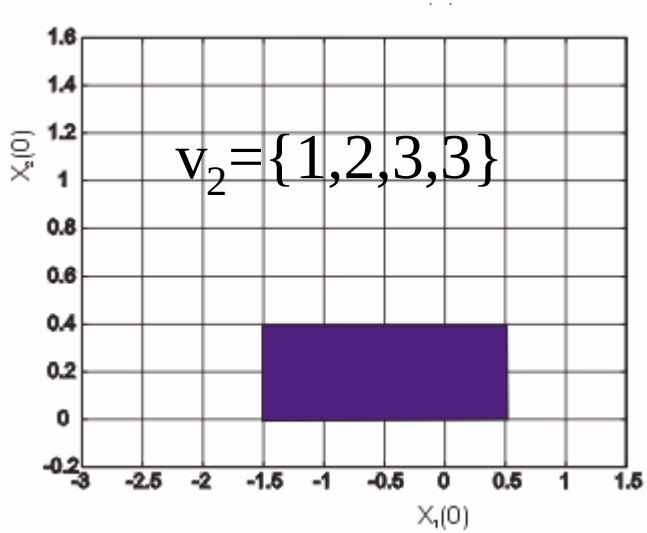
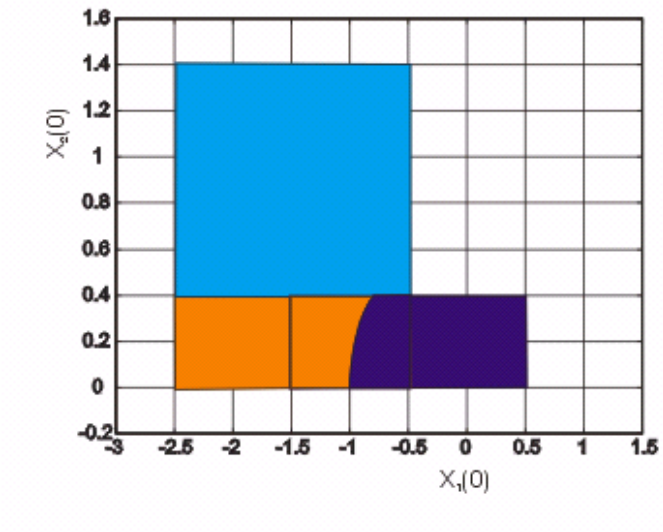






Computing Optimal Control Law

The proofs are constructive but based on the enumeration of all possible switching sequences of the hybrid system.

More efficient methods than enumeration are necessary

mp-MIQP Algorithm

(Borrelli, Baotic, Bemporad, Morari, 2002)

Solver based on

- Dynamic programming recursion
- Multiparametric quadratic program solver (mp-QP)
- Basic polyhedral manipulation

  (intersection and union)
- Comparison of value functions over polyhedra
- Post processing

Optimal Control Problem

Problem

$$J^*(x(0)) \quad := \quad \min_{U^N} J(U^N, x(0))$$
$$\text{s.t.} \begin{cases} x(t+1) = f_{PWA}(x(t), u(t)) \\ x(T) \in X^f \end{cases}$$

where

$$J(U^N, x(0)) := \|Px(N)\|_2 + \sum_{k=0}^{N-1} \|Qx(k)\|_2 + \|Ru(k)\|_2$$

and

$$x(k+1) = f_{PWA}(x(k), u(k)) \quad \Longleftrightarrow \quad \begin{aligned} x(t+1) \quad &= \quad A_i x(t) + B_i u(t) + f_i \\ &\text{if } [x(t), u(t)] \in X_i \end{aligned}$$

Closed Loop Solution via Dynamic Programming

Equivalent problem

$$J_j^*(x(j)) \quad := \quad \min_{u(j)} \left\{ \|Qx(j)\|_2 + \|Ru(j)\|_2 + J_{j+1}^*(f_{PWA}(x(j),u(j))) \right\}$$
$$\text{s.t.} \quad f_{PWA}(x(j),u(j)) \in X^{j+1}$$

Cost to go

for $j=N$-1,...,0, with boundary conditions

$$X^N = X^f$$
$$J_N^*(x(N)) = \|Px(N)\|_2$$

where X^j is the set of initial states for which problem is feasible

$$X^j = \left\{ x \in R^n \mid \exists u,\ f_{PWA}(x,u) \in X^{j+1} \right\}$$

Challenge

- Challenge
  - Sub-problems are defined over non-convex regions
- Method
  - Multiparametric quadratic program solver (mp-QP)
  - Polyhedral manipulation (intersection and union)
  - Comparison of value functions over polyhedra
- Result
  - Exact, globally optimal PWA feedback control law

Example

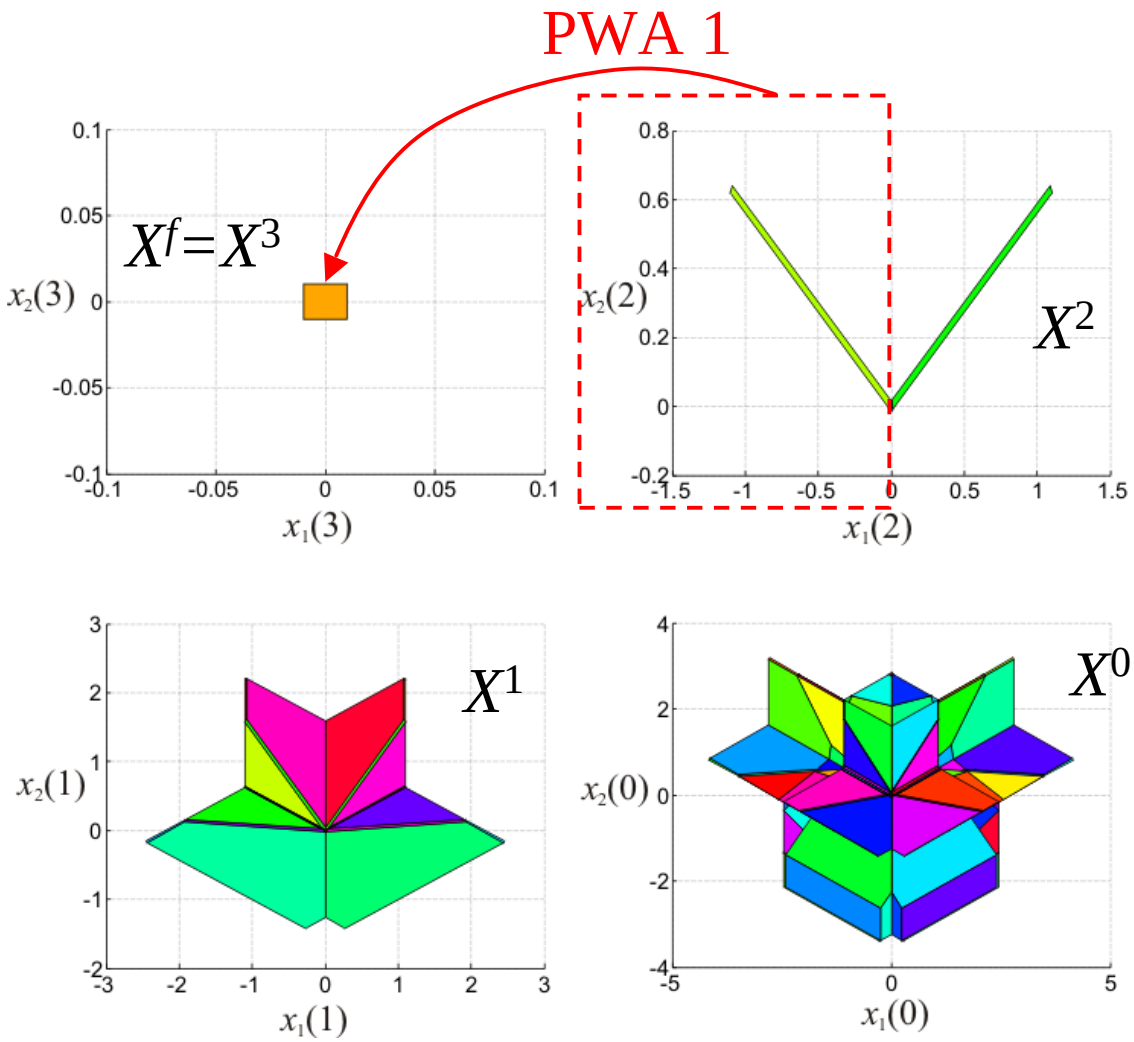
$$\min_U \qquad ||Px(N)||_p + \sum_{k=0}^{N-1} ||Qx(k)||_p + ||Ru(k)||_p$$

$$\text{subj. to} \begin{cases} x(t+1) &= 0.8\begin{bmatrix} \cos a(t) & -\sin a(t) \\ \sin a(t) & \cos a(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \\ a(t) &= \begin{cases} p/3 & \text{if} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \geq 0 \\ -p/3 & \text{if} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) < 0 \end{cases} \\ x(t) &\in \begin{bmatrix} -10, & 10 \end{bmatrix} \times \begin{bmatrix} -10, & 10 \end{bmatrix} \\ u(t) &\in \begin{bmatrix} -1, & 1 \end{bmatrix} \end{cases}$$

$$p = 2, \quad N = 3, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Dynamic Programming - Illustration

N=3, 2 PWA dynamics



Post processing

After every step of the Dynamic Program

- Remove unnecessary regions

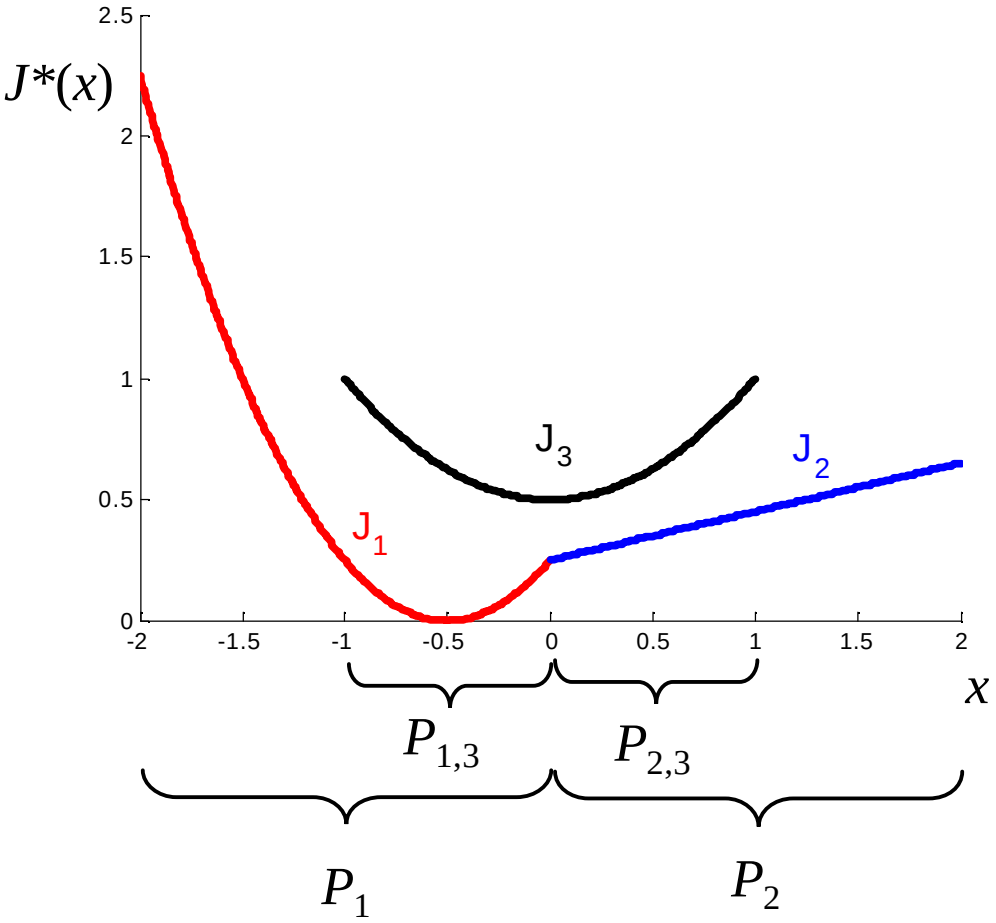
  (Intersect, Compare, Union)

At the end of computation

- Join regions with the same control law
- Store regions in an ordered fashion

Union of regions

- Intermediate stage simplification



$$J_1(x) \le J_3(x), \quad \text{for } x \in P_{1,3}$$

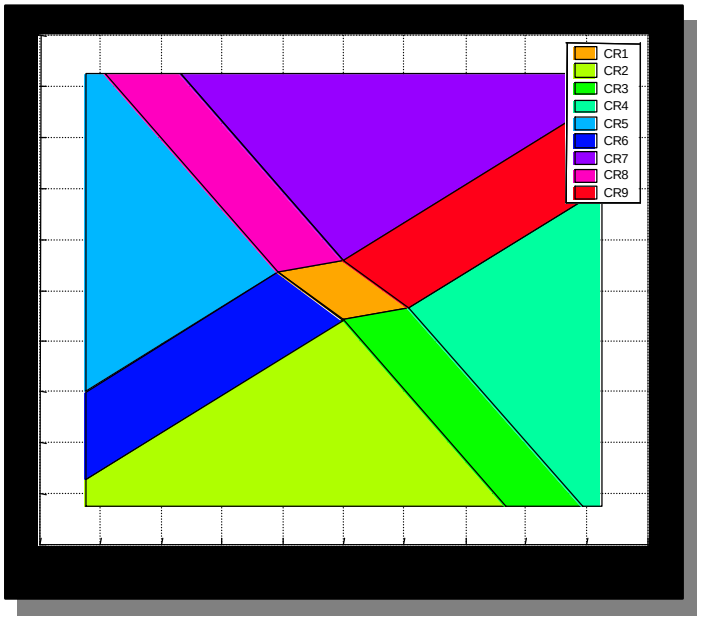
$$J_2(x) \le J_3(x), \quad \text{for } x \in P_{2,3}$$

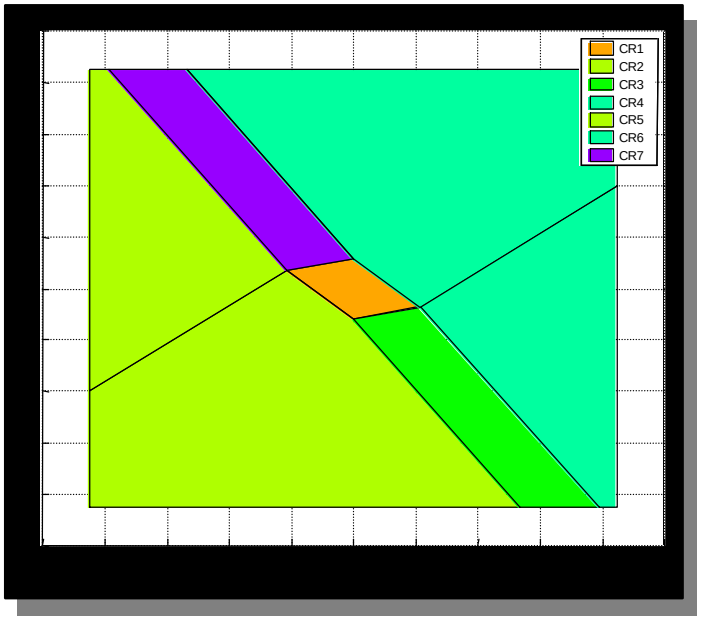
$$P_{1,3} \cup P_{2,3} = P_3$$

No need for P_3

Union of regions

- Simplifying final solution

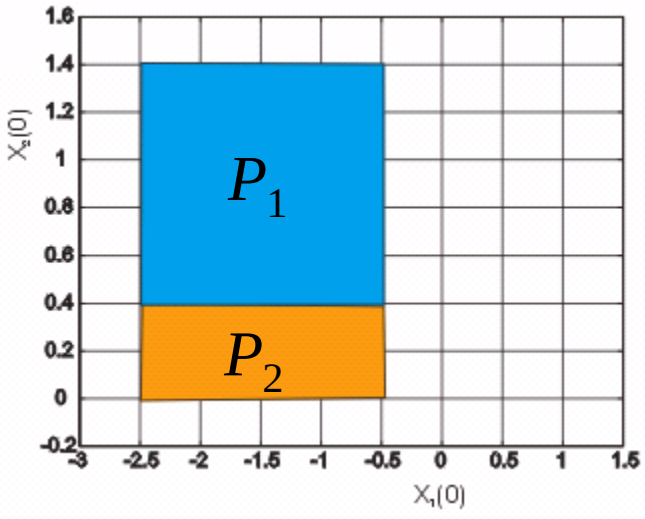


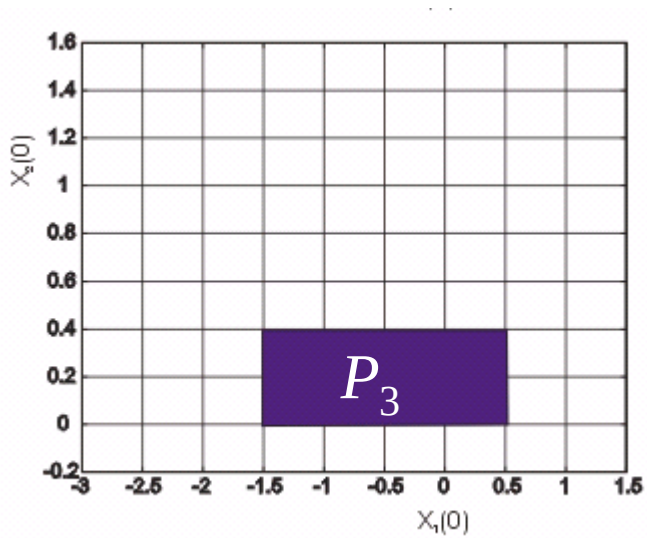


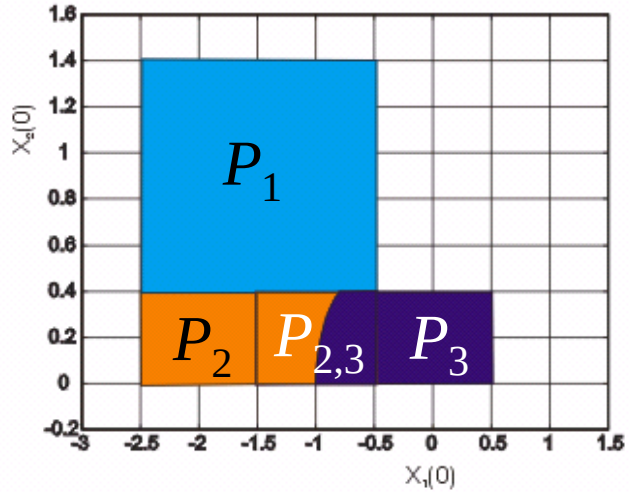
$$U^*(x) = \{u_0^*, u_1^*, \ldots, u_{N-1}^*\} \qquad u = u_0^* = f(x)$$

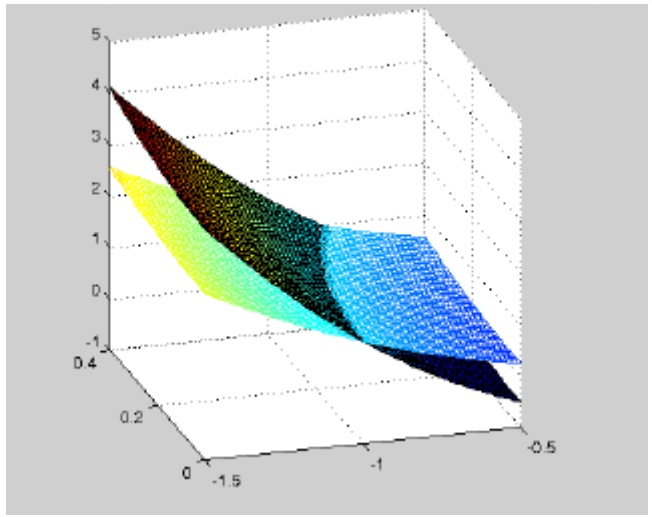
Regions where the first component of the solution is the same can be joined (when their union is convex). (Bemporad, Fukuda, Torrisi, *Computational Geometry*, 2001)

Intersection of regions





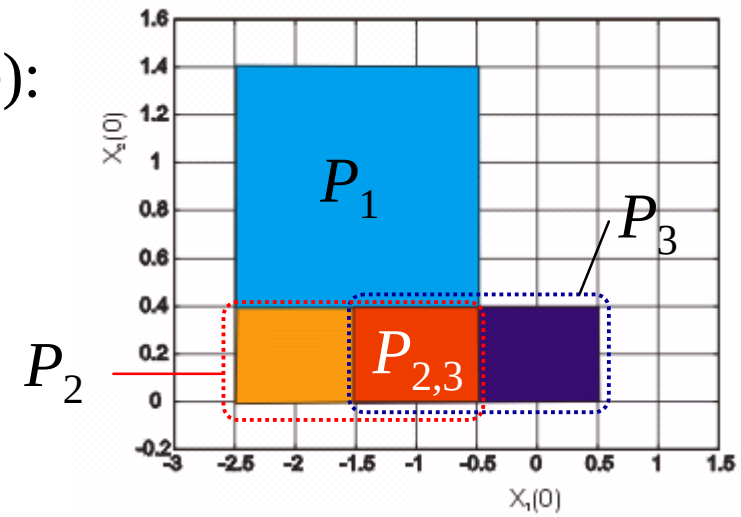




Intersection of regions

Solution (in the ordered region sense):

$$\{P_{2,3}, P_1, P_2, P_3\}$$

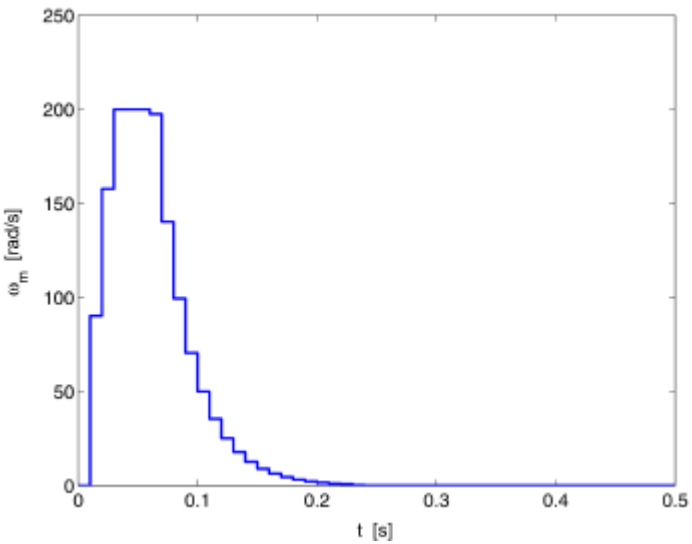


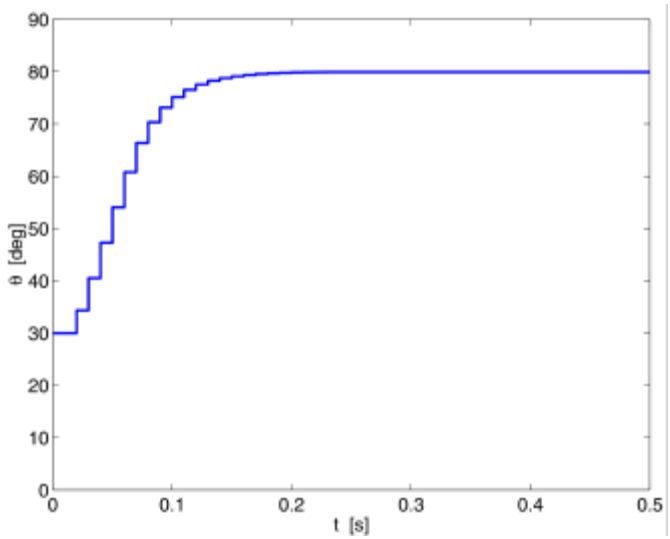
$$u^*(x) = \begin{cases} u_2^*(x) & \text{if } x \in P_{2,3} \text{ and } J_2^*(x) \le J_3^*(x) \\ u_3^*(x) & \text{if } x \in P_{2,3} \text{ and } J_3^*(x) \le J_2^*(x) \\ u_1^*(x) & \text{if } x \in P_1 \\ u_2^*(x) & \text{if } x \in P_2 \\ u_3^*(x) & \text{if } x \in P_3 \end{cases}$$

Throttle Results

$$i_a(k) \in [-2,\ 2],\ w_m(k) \in [-200,\ 200]$$
$$q(k) \in [11,\ 90],\ u(k) \in [-5,\ 5]$$

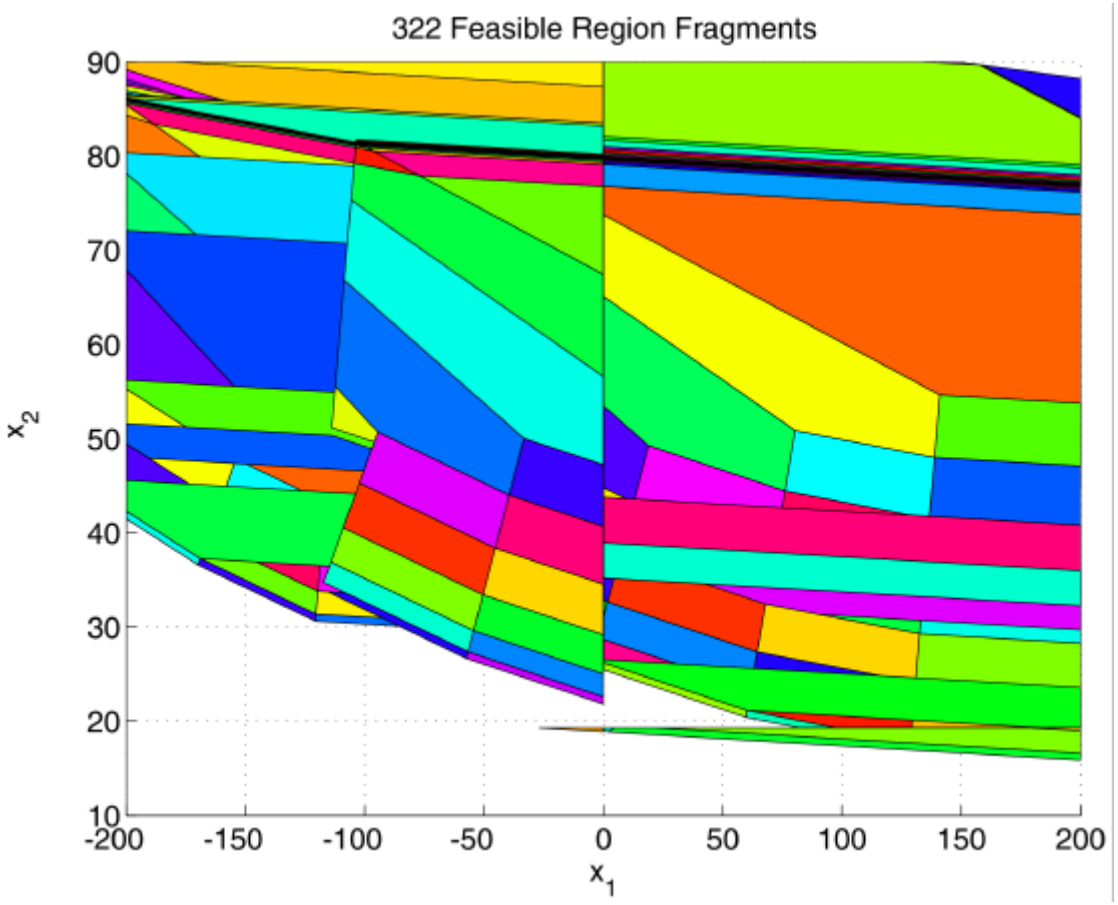
$$T_s = 10ms,\ N = 10,\ Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix},\ R = 0.1,\ P = Q,$$
$$X^f = [-0.1\ 79.9] \times [0.1\ 80.1],\ x_e = [0\ 80]^T$$





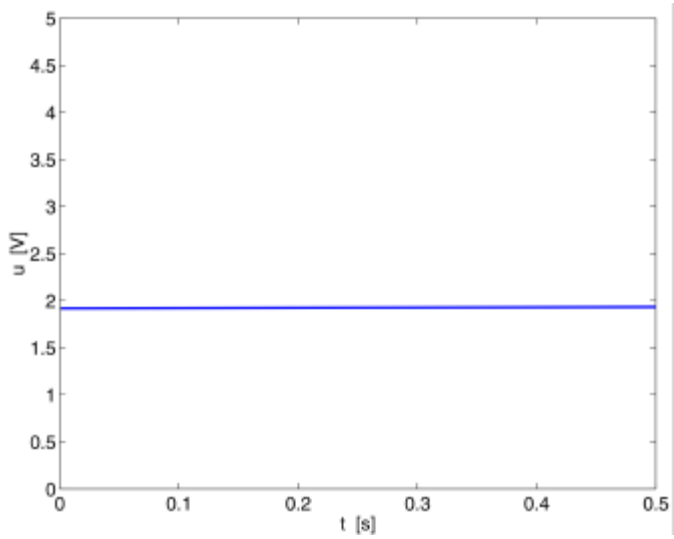
Throttle Results

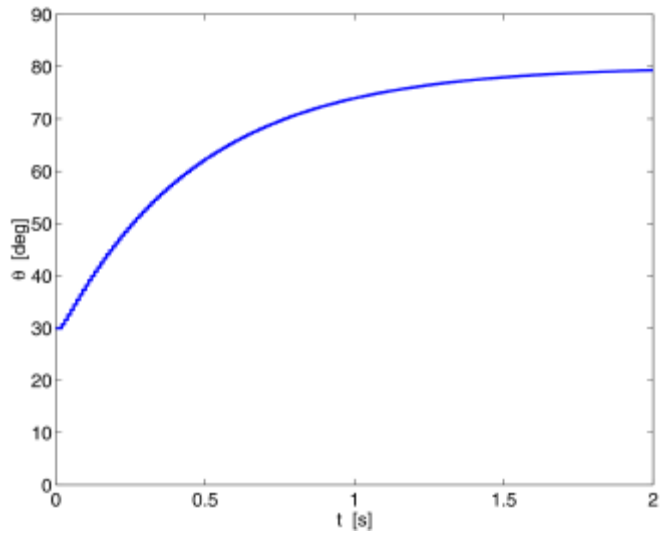
Partition of the state space describing PWA optimal control law



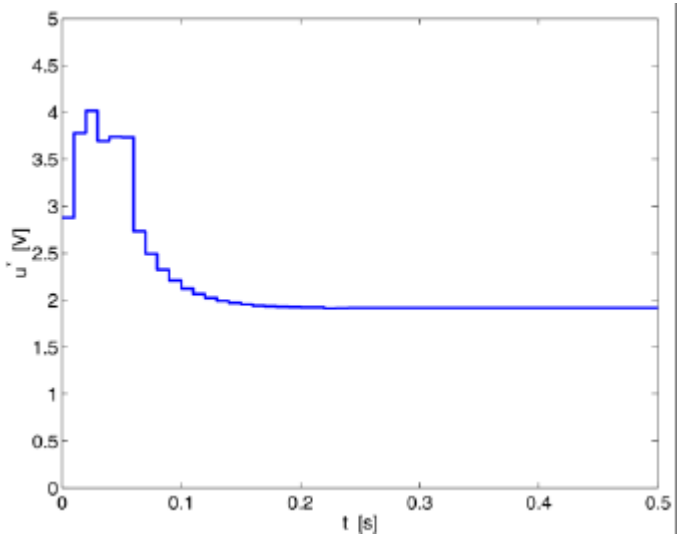
Throttle Results

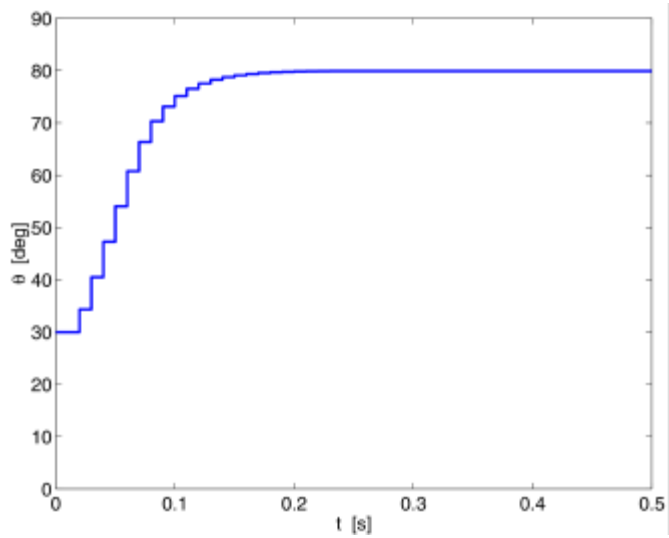
Open loop





Closed loop





Conclusions

- Electronic throttle modeled as a PWA system
- Finite-time constrained optimal control problems based on quadratic performance index for discrete time PWA systems
- mp-MIQP Algorithm
  - Dynamic Programming recursion
  - Intersection of regions
  - Comparison of value functions
  - Ordering of regions
- Receding Horizon control can be implemented by using a simple look-up table
- Results on the throttle control

THE END