Observability of Piecewise-Affine Hybrid Systems

Pieter Collins (CWI)
Jan H. van Schuppen (CWI)

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Outline

- Motivation.
- Observability of PAHS. Framework.
- Sufficient conditions for observability.
- Necessary conditions for observability.
- Concluding remarks.

Motivation for investigating observability of PAHS

- Paper by Andrea Balluchi et al. on observers.
- Papers on observability of hybrid systems presented at HSCC.2003 in April 2003.
- Observability is sufficient condition for existence of observers.
- Observability used in realization theory.
- Comparison observability and reachability.

Framework for observability of PAHS

Def. Piecewise-affine hybrid system (PAHS, CT, Time-invariant)

$$
\begin{array}{rcl}
Q & & \text{finite state set, } U \subseteq \mathbb{R}^m,\ Y \subseteq \mathbb{R}^p, \text{ polyhedral sets,} \\
X(q) & \subseteq & \mathbb{R}^{n(q)},\ \forall q \in Q, \text{ closed polyhedral sets,} \\
\dot{x}_q(t) & = & A(q)x_q(t) + B(q)u(t) + a(q),\ x_q(t_o) = x_q^+, \\
y(t) & = & C(q)x_q(t) + D(q)u(t) + c(q), \\
e & \in & E_{in}, \text{ input event, or} \\
e & \in & E_{cd}, \text{ if } x(t_1) \in G_q(e) \subset \partial X(q), \\
 & & \text{event generated by continuous dynamics; then transition,} \\
q^+ & = & f(q^-, e),\ q_0, \\
x_{q^+}^+ & = & A_r(q^-, e, q^+)x_{q^-}^- + b_r(q^-, e, q^+) : G_{q^-} \subseteq X(q^-) \rightarrow X_0(q^+).
\end{array}
$$

Assumptions: (1) Finite number of events at any time.

(2) Finite number of events on any finite interval (non-Zenoness).

Framework for observability of PAHS

Remark For a dynamic systems consider the map,

$$\text{(initial state,input trajectory)} \ \mapsto \ \text{output trajectory.}$$

Observability holds if the initial state is uniquely determinable from the input-output trajectories.
Note dependence on input trajectory as in nonlinear systems.
Restriction first to Case ‘No input’.

Distinguish

- **Observability**: Injectiveness of the map

$$x(t_0) \ \mapsto \ y : T \to Y, \ \ T = [t_0, t_1], \ t_1 \in \mathbb{R} \cup \{\infty\}.$$

- **Co-Observability or reconstructibility**: Injectiveness of the map

$$x(t_1) \ \mapsto \ y : T \to Y, \ \ T = [t_0, t_1], \ t_1 \in \mathbb{R} \cup \{\infty\}.$$

Framework for observability

Concepts of observability proposed by E.D. Sontag [Sontag, 1978].

1. Observability 1. Injectiveness of the map,

$$x(t_0) \;\mapsto\; y : T \to Y, \;\; T = [t_0, t_1],$$

2. Observability 2. Existence of an observer with a finite-dimensional state space.

Remarks

- Relations between observability concepts. [Sontag, 1978].
- For polynomial systems Observability 2 implies Observability 1.
- Condition for Observability 2 of polynomial systems related to fin.-dim. of observation algebra.
- Observability 2 also used for systems on manifolds and for stochastic systems described by stochastic differential equations.

Framework for observability

What are the observations of PAHS? Recall restriction to no input case.

- **Partly observed discrete-event system:**
  Event set E, subset of observable events $E_o \subseteq E$,
  natural projection $P : E^* \to E_o^*$,
  $P(e) = e$, if $e \in E_o$, $= \epsilon$, if $e \in E \backslash E_o$. $e_o(k) = P(e(i))$

- **Observed timed-event sequence,** either of finite or of infinite length,

$$\{(t_k, e_o(k)) \in T \times E_o,\ k \in \mathbb{Z}_n \text{ or } k \in \mathbb{Z}_+\},$$

$$t_0 \leq t_1 \leq \ldots \leq t_k \leq \ldots$$

- **Sequence of output trajectories:**

$$\{y : T_k \to Y,\ k \in \mathbb{Z}_n \text{ or } k \in \mathbb{Z}_+\}, \quad T_k = [t_{k-1}, t_k].$$

Framework for observability

Case summary.

$$
\begin{aligned}
X(q) &= \mathbb{R}^{n_q}, \quad \text{not polytope,} \\
\dot{x}_q(t) &= A_q x_q(t),\ x_q(t_0) = x_{q,0}, \\
y(t) &= C_q x_q(t), \\
q^+ &= f(q^-, e(i)),\ q_0, \\
x^+_{q^+} &= H(q^+, q^-) x^-_{q^-}, \\
& \quad H(q^+, q^-) : G_{q^-} \subseteq X(q^-) \rightarrow X_0(q^+) \subseteq X(q^+), \\
e_o(k) &= P(e(i)), \\
& \quad \{(t_k, e_o(k), y_k), k \in Z_+\},\ \text{observations,} \\
& \quad y_k : T_k = [t_{k-1}, t_k) \rightarrow Y = \mathbb{R}^p.
\end{aligned}
$$

Notation

Observability matrix of $(A, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{p \times n}$,

$$\mathrm{Obsm}(A, C) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \in \mathbb{R}^{pn \times n}.$$

(A, C) called **observable pair** if $\mathrm{rank}(\mathrm{Obsm}(A, C)) = n$.

Transition map of continuous and discrete transition

$$S : X(q_1) \to X(q_2),$$

$$S(q_2, q_1) \quad = \quad H(q_2, q_1) \exp(A(q_1)(t_2 - t_1)).$$

Def. Consider the PAHS system formulated above.
The system is said to be **observable**
if the following map is injective:

$$(q_0, x_{q_0,0}) \mapsto \{(t_k, e_o(k), y_k), k \in Z_+\}.$$

Problem Characterize observability of a PAHS system.

Remarks

(E.D. Sontag, 1995) discusses the computational complexity of a reachability problem for a discrete-time PL system.
Relevance for observability indicated.
Problem posed in this lecture: for continuous-time PAHS and for a strict subclass.
Thus, decidability of Characterization of observability is currently an open problem.

Sufficient conditions for observability of PAHS

Theorem (Sufficient condition 1, not necessary condition)
A PAHS as considered above is observable if

1. Condition 1: **Observability of discrete state from continuous output:** for all $q_1, q_2 \in Q$,

$$\begin{aligned} & x_{q_1,0} \in X_0(q_1),\ x_{q_2,0} \in X_0(q_2), \\ & C(q_1)A(q_1)^k x_{q_1,0} = C(q_2)A(q_2)^k x_{q_2,0},\ \forall k \in \mathbb{N}, \\ \Rightarrow \quad & q_1 = q_2. \end{aligned}$$

   Checkable equivalent condition stated below.

2. for all $q \in Q$, $(A(q), C(q))$ is an observable pair.

Proof

(1)Estimation of the discrete state - First step.

A discrete-event observer using the observed timed-event sequence

$$(t, s) = (t_k, e_o(k)),\ k \in \mathbb{Z}_r)$$

produces after each event a subset of the discrete state set compatible with the observed timed-event sequence,

$$Q_{obs}(t, s) = \left\{ \begin{array}{l} q \in Q | \exists \{(t_i, v_i), i \in \mathbb{Z}_r\}, \\ \exists q_0 \in Q,\ q = f(q_0, v),\ P(v) = s \end{array} \right\}.$$

Note that the subset $Q_{obs}(t, s)$ may contain two or more discrete states.

Proof (Continued)

(2) Estimation of the discrete state - Second step.

Let $q_1, q_2 \in Q_{obs}((t, s))$. If observed continuous output yields,

$$
\begin{aligned}
& C(q_1)\exp(A(q_1)(t-t_0))x_{q_1,0} = C(q_2)\exp(A(q_2)(t-t_0))x_{q_2,0} \\
& \forall t \in [t_0, t_1), \\
\Rightarrow \quad & C(q_1)A(q_1)^k x_{q_1,0} = C(q_2)A(q_2)^k x_{q_2,0},\ \forall k \in \mathbb{N}, \\
\Rightarrow \quad & \text{(Condition 1) } q_1 = q_2.
\end{aligned}
$$

Else can distinguish q_1 and q_2.

(1) and (2) imply that the current discrete state can be determined uniquely.

Proof (Continued)

(3) From Step (2) follows that $q \in Q$ known.

Realization theory of FDLS and $(A(q), C(q))$ an observable pair,

imply that $x_{q,0}$ can be determined uniquely from, for all $t \in T_1 = [t_0, t_1)$,

$$\begin{aligned} y(t) &= C(q) \exp(A(q)(t - t_0)) x_{q,0}, \\ \begin{pmatrix} y(s) \\ \dot{y}(s) \\ \vdots \\ y^{(n(q)-1)}(s) \end{pmatrix} \Big|_{s=t_0} &= \text{Obsm}(A(q), C(q)) x_{q,0}. \end{aligned}$$

Proposition Consider a PAHS.

Condition 1 is equivalent to

(1) $\mathrm{spec}(A(q_1)) \cap \mathrm{spec}(A(q_2)) = \emptyset$, or

(2) when realizations,

$$C(q_1)A(q_1)^k x_{q_1,0} = C(q_2)A(q_2)^k x_{q_2,0}, \ \forall k \in \mathbb{N},$$

reduced to common spectrum, including multiplicities, then

$$\begin{aligned} & \nexists x_{q_1,0} \in X_0(q_1), \quad \nexists x_{q_2,0} \in X_0(q_2), \text{ not both zero,} \\ & \mathrm{Obsm}(A_r, C_r)(x_{q_1,0} - x_{q_2,0}) = 0, \\ \Leftrightarrow \quad & \ker(\mathrm{Obsm}(A_r, C_r)) \cap (X_0(q_1) - X_0(q_2)) = \{0\}. \end{aligned}$$

Remarks

(a) If,

$$\forall q_1, q_2 \in Q,\ \forall x_{q_1} \in X(q_1),\ \forall x_{q_2} \in X(q_2),$$
$$C(q_1)x_{q_1} \neq C(q_2)x_{q_2},$$

then the continuous output signals every discrete transition. Otherwise it signals this only for particular discrete states and continuous states.

(b) If

$$\forall q_1, q_2 \in Q,\ \forall x_{q_1} \in X(q_1),\ \exists k \in \mathbb{N},$$
$$C(q_1)A(q_1)^k x_{q_1} \neq C(q_2)A(q_2)^k H(q_2, q_1)x_{q_1},$$

then the existence of a transition $q_1 \mapsto q_2$ can be determined from the continuous output.

Example 1

$$\begin{aligned}
Q &= \{q_1, q_2\}, \\
X(q_1) &= \mathbb{R}^2,\ X_0(q_1) = \operatorname{span}\{e_1 + e_2\}, \\
\dot{x}_{q_1}(t) &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x_{q_1}(t),\ x_{q_1}(t_0) = x_{q_1,0} \in X_0(q_1), \\
y(t) &= \begin{pmatrix} 1 & 3 \end{pmatrix} x_{q_1}(t), \\
X(q_2) &= \mathbb{R},\ X_0(q_2) = \{1\}, \\
\dot{x}_{q_2}(t) &= 3x_{q_2}(t),\ x_{q_2}(t_0) = 1 \in X_0(q_2), \\
y(t) &= x_{q_2}(t);
\end{aligned}$$

$$
\begin{aligned}
E &= \{e(2,1), e(1,2)\}, \quad E_o = \{e(1,2)\}, \\
G_{q_1}(e(2,1)) &= (-\infty, 1] \times (-\infty, 0],\ G_{q_2}(e(1,2)) = [5, \infty), \\
q_2 &= f(q_1, e(2,1)),\ q_1 = f(q_2, e(1,2)), \\
x_{q_2}^{+} &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_{q_1}^{-},\ x_{q_1}^{+} = 1.
\end{aligned}
$$

The PAHS is observable because,

(1) $\mathrm{spec}(A(q_1) \cap \mathrm{spec}(A(q_2) = \emptyset,$

(2) $\forall q \in Q,\ (A(q), C(q))$ is observable pair.

Example 2 As Example 1 but with the changes,

$$\dot{x}_{q_1}(t) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} x_{q_1}(t).$$

The PAHS is observable because,

(1)

$$\mathrm{spec}(A(q_1)) \cap \mathrm{spec}(A(q_2)) \neq \emptyset,$$

$$\begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_{q_1,0} \neq 3^k 1.$$

(2) For all $q \in Q$, $(A(q), C(q))$ is an observable pair.

Sufficient condition for observability of PAHS

Theorem (Sufficient condition 2)

A PAHS is observable if

(1) Observability of discrete state from continuous trajectory holds for all discrete states in observer states with two or more discrete states:

$$
\begin{aligned}
& |Q_{obs}(t_0, s)| \geq 2, \\
\Rightarrow \quad & \forall q_1, q_2 \in Q_{obs}(t, \epsilon, q), \text{ Condition 1 holds.}
\end{aligned}
$$

Continued on next slide.

(2) For all $q \in Q$

if there exists $x_{q,0} \in X_0(q)$ such that no guard is reachable

then $(A(q), C(q))$ must be an observable pair;

else (if for all $x_{q,0} \in X_0(q)$ a guard is reachable)

$x_{q,0}$ must be observable from the resulting sequence of continuous output trajectories for known sequence of discrete states visited.

Details in next proposition.

Proposition Observability of continuous state from sequence of continuous outputs for known sequence of discrete states.

(a) Case of two discrete states and transition $q_1 \mapsto q_2$.
Observability of continuous state in case of sequence of two discrete states holds
if and only if

$$\begin{aligned} \begin{pmatrix} \text{Obsm}(A(q_1), C(q_1)) \\ \text{Obsm}(A(q_2), C(q_2))S(q_2, q_1) \end{pmatrix} = n(q_1); \\ S(q_2, q_1) \quad = \quad H(q_2, q_1) \exp(A(q_1)(t_1 - t_0)). \end{aligned}$$

Proposition (Continued)

(b) Case of a sequence with three discrete states $q_1 \mapsto q_2 \mapsto q_3$, observability of continuous state holds if and only if,

$$\text{rank} \begin{pmatrix} \text{Obsm}(A(q_1), C(q_1)) \\ \text{Obsm}(A(q_2), C(q_2)) S(q_2, q_1) \\ \text{Obsm}(A(q_3), C(q_3)) S(q_3, q_2) S(q_2, q_1) \end{pmatrix} = n(q_1).$$

Remark Observability of PAHS dual to reachability of PAHS as developed in (JHvS, 1998).

Example 3

$$
\begin{aligned}
Q &= \{q_1, q_2\}, \\
X(q_1) &= \mathbb{R}^2,\ X_0(q_1) = \{10\} \times \mathbb{R}, \\
\dot{x}_{q_1}(t) &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x_{q_1}(t),\ x_{q_1}(t_0) = x_{q_1,0} \in X_0(q_1), \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_{q_1}(t), \\
X(q_2) &= \mathbb{R}^2,\ X_0(q_2) = \{30\} \times \mathbb{R}, \\
\dot{x}_{q_2}(t) &= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} x_{q_2}(t),\ x_{q_2}(t_0) = x_{q_2,0} \in X_0(q_2), \\
y(t) &= \begin{pmatrix} 5 & 6 \end{pmatrix} x_{q_2}(t),
\end{aligned}
$$

Example 3 (Continued)

$$\begin{aligned} E &= \{e(2,1), e(1,2)\},\ E_o = E, \\ G(q_1) &= (-\infty, 1] \times \mathbb{R},\ G(q_2) = [50, \infty) \times \mathbb{R}, \\ q_2 &= f(q_1, e(2,1)),\ q_1 = f(q_2, e(1,2)), \\ x_{q_2}^{+} &= x_{q_1}^{-},\ x_{q_1}^{+} = x_{q_2}^{-}. \end{aligned}$$

The PAHS is observable because

(1) discrete states are distinguishable,

Example 3 (Continued)

(2) $(A(q_1), C(q_1))$ is not an observable pair,
from all $x_{q_1,0} \in X_0(q_1)$
the guard $G_{q_1} = (-\infty, 1] \times \mathbb{R}$ is reachable, and

$$n(q_1) = 2 \quad = \quad \mathrm{rank} \begin{pmatrix} \mathrm{Obsm}(A(q_1), C(q_1)) \\ \mathrm{Obsm}(A(q_2), C(q_2)) S(q_2, q_1) \end{pmatrix},$$

$$\mathrm{Obsm}(A(q_1), C(q_1)) \quad = \quad \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$$

$$\mathrm{Obsm}(A(q_2), C(q_2)) S(q_2, q_1)$$

$$= \quad \begin{pmatrix} 5 & 6 \\ 15 & 24 \end{pmatrix} \exp(\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} (t_1 - t_0))$$

Necessary condition for observability

Conjecture (Necessary condition)

If the considered PAHS is observable then:

(1) Observability of discrete states from continuous trajectory holds for all discrete states in observer states with two or more discrete states;

(2) For all $q \in Q$

if there exists $x_{q,0} \in X_0(q)$ such that no guard is reachable

then $(A(q), C(q))$ must be an observable pair;

else (if for all $x_{q,0} \in X_0(q)$ a guard is reachable)

$x_{q,0}$ must be observable from the resulting sequence of continuous output trajectories for known sequence of discrete states.

Concluding remarks
Results

- Framework for observability of PAHS.
- Sufficient condition for observability of PAHS.
- Necessary condition for observability of PAHS.

Further research

- Necessary and sufficient condition.
- Case with input and with polytopes.
- Existence of observer in terms of observation algebra.