Chaotic Dynamics in Hybrid Systems

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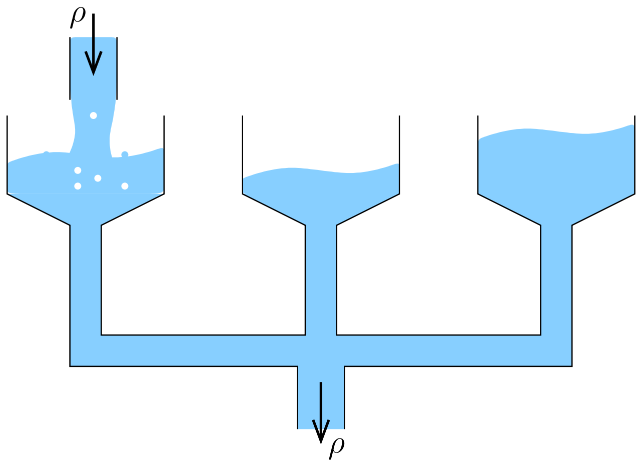
Outline

- Some chaotic hybrid systems
- A formalism for hybrid systems
  - Hybrid time space
  - Skorohod topology
  - Upper semicontinuity
- Properties of upper-semicontinuous systems
- Example-filling tanks
- Conclusions and further research

Some chaotic hybrid systems

- Switched arrival systems



  - Switch whenever a tank empties.

[Ref: Chase, Serrano & Ramadge. *IEEE Trans. Automatic Control*, 38(1) 70–83, 1993]

- Impact oscillators

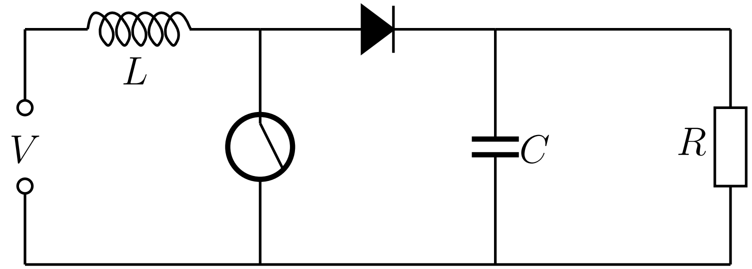
$$\ddot{x} + \zeta \dot{x} + x = \cos(\omega t), \quad x < d;$$

$$\dot{x} \mapsto -\lambda \dot{x}, \quad x = d.$$

[Ref: Budd, in *Nonlinear mathematics and its applications*, CUP, 1996]

Some chaotic hybrid systems

- Boost power converter



- Inductor current I and capacitor charge Q.
- Switch open:

$$I \geqslant 0: \quad L\frac{dI}{dt} = E - \frac{Q}{C}, \quad \frac{dQ}{dt} = I - \frac{Q}{RC}.$$

$$\text{or } Q \geqslant EC: \quad I = 0, \quad \frac{dQ}{dt} = -\frac{Q}{RC}.$$

- Switch closed: $Q \geqslant 0$

$$L\frac{dI}{dt} = E; \quad \frac{dQ}{dt} = -\frac{Q}{RC}$$

[Ref: Banerjee et al. *IEEE Trans. Circuits Systems*, I47(3), 389–394, 2000]

Fundamental issues

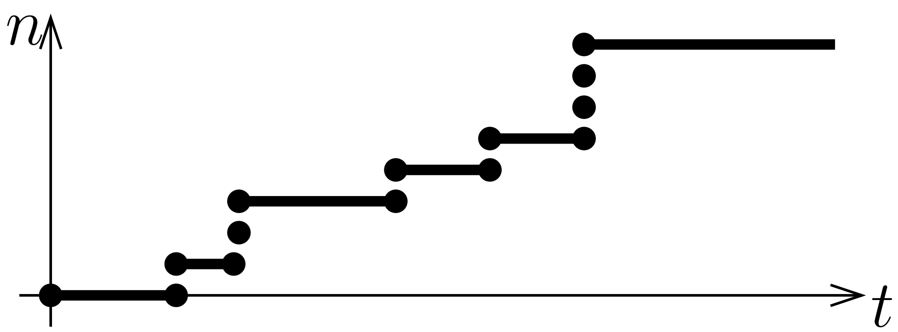
- Existence of limiting trajectories.
- Zeno executions.
- Statistical properties — invariant measures and ergodic theory.
- Orbit structure — symbolic dynamics.
- Parameter dependence — bifurcation theory.

A framework for hybrid systems

- A hybrid system $\mathcal{H}$ without inputs with state space X is described by
  - A vector field $F : X \to TX$
  - A reset condition $f : X \to \mathcal{P}(X)$
- We allow vector fields which are not everywhere defined.
- At any point x, there may be many possible resets, or none at all.

Hybrid time sets

- An *event time sequence* $(t_n)_{n=1}^N$ with $N \in \mathbb{Z}^+ \cup \{\infty\}$ is a monotone sequence $t_n \leqslant t_{n+1}$.
- The *hybrid time set* for the event time sequence (t_n) is the set $\tau = \{(n,t) : t \in [t_n, t_{n+1}]\} \subset \mathbb{Z}^+ \times \mathbb{R}^+$.



- A hybrid time set gives a natural domain for a trajectory of a hybrid system..
- Note that we allow more than one reset to occur at a given time.

[Ref: Lygeros et al.]

A trajectory of a hybrid system

- A *trajectory* or *execution* of $\mathcal{H}$ is a function $x : \tau \to X$ such that

$$\dot{x}(n,t) = F(x(n,t)) \text{ for } t \in (t_n, t_{n+1});$$

$$x(n,t_n) \in f(x(n-1,t_n)).$$

- Note that
  - A reset *must* happen if there is no further continuous trajectory; otherwise it *may* occur.
  - There may be more than one possible trajectory for a given initial condition.

Continuous and discrete dynamics

- The *continuous dynamics* of a trajectory x is given by the function $x : \mathbb{R}^+ \to X$ with
$$x(t) = x(n,t) \text{ for } t \in [t_n, t_{n+1}).$$

- The *discrete dynamics* of x is given by the sequence $(x_n)_{n=0}^N$ with
$$x_n = x(n, t_n).$$

- The *return map* $r : X \to \mathcal{P}(X)$ is given by $x_1 \in r(x_0)$ if there is a trajectory x with
$$x(0,t_0) = x_0 \text{ and } x(1,t_1) = x_1$$

Hybrid Skorohod topology

- The natural topology for the space of trajectories is the *compact-open hybrid Skorohod topology*.
- A trajectory $\tilde{x}$ is (N, T, ϵ)-close to x if there is a map $h : \tau \to \tilde{\tau}$ such that whenever $n \leqslant N$ and $t \leqslant T$, then
  - $h(n, t) = (n, \tilde{t})$ with $|t - \tilde{t}| < \epsilon$.
  - $d(x(n, t), \tilde{x}(h(n, t))) < \epsilon$.
- The (N, T, ϵ) neighbourhoods of trajectories form a basis of open sets for the topology.
- An equivalent metric description of the topology can also be formulated.

Upper semicontinuity

- A set-valued function $f : X \to \mathcal{P}(Y)$ is *upper-semicontinuous* if $f(K)$ is compact whenever K is compact.
- If f is upper-semicontinuous, and $x_i \to x_\infty$ and $y_i \to y_\infty$ are sequences with $y_i \in f(x_i)$, then also $y_\infty \in f(x_\infty)$.
- A hybrid system $\mathcal{H}$ is *upper-semicontinuous* if its reset map f is upper semicontinuous.
- An example of an upper-semicontinuous hybrid system is a system with events e, and continuous reset maps $f_e : X_e \to X$ defined on closed subsets X_e of X.

Limiting trajectories

- The fundamental property of upper-semicontinuous hybrid systems is that limits of trajectories are also trajectories:
- **Theorem** *Let $\mathcal{H}$ be an upper-semicontinuous hybrid system with compact state space, and ξ_n a sequence of trajectories of $\mathcal{H}$ converging to ξ_∞ in the compact-open hybrid Skorohod topology. Then ξ_∞ is a trajectory of $\mathcal{H}$.*
- It follows that if $\mathcal{H}$ has an upper-semicontinuous reset map, then its return map is also upper-semicontinuous.

Zeno executions

- An execution x of a hybrid system is *Zeno* if infinitely many transitions occur in finite time T. (i.e. $\lim_{n\to\infty} t_n = T < \infty$)
- An execution x is *instantaneously Zeno* if $t_n = 0$ for all $n \in \mathbb{N}$. (i.e. infinitely many resets can occur without any continuous dynamics.)
- A hybrid system is *uniformly non-Zeno* if there is an integer N and a time T such that for any execution x, there are at most N discrete transitions in any time interval of length T.

Zeno hybrid systems

- An instantaneously Zeno execution can be thought of as a chattering behaviour, and is usually due to a modelling error.
- If a hybrid system is uniformly non-Zeno (and non-blocking), then all trajectories are defined for all $t \in R^+$.
- The following result shows that an upper-semicontinuous hybrid system is either very well-behaved or very nasty:
- **Theorem** *An upper-semicontinuous hybrid system with compact state space is either uniformly non-Zeno or has an instantaneously Zeno execution.*

Invariant measures

- Try to give a probabilistic description of a hybrid system by finding an invariant probability measure for its return map.
- If $f : X \to X$ is a single-valued map, a measure μ on X is *invariant* under f if $\mu(f^{-1}(A)) = \mu(A)$ for all measurable sets A.
- Any continuous map on a compact metric space has an invariant probability measure. Does an analogous result hold for upper-semicontinuous maps?

Shift-invariant measures

- The *orbit space* Σ of a map $f : X \to \mathcal{P}(X)$ is the set of sequences (x_i) with $x_{i+1} \in f(x_i)$.
- The *shift map* on Σ is the map σ such that

$$\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$$

- If f is upper-semicontinuous, then Σ is compact. Since σ is continuous, it has an invariant measure μ_σ.
- The measure μ on X given by $\mu(A) = \mu_\sigma(\pi^{-1}(A))$, where $\pi : \Sigma \to X$ is projection onto the first factor, is a *shift-invariant measure* for f.

Invariant measures for hybrid systems

- The return map for an upper-semicontinuous hybrid system is also upper semicontinuous. Therefore:
- **Theorem** *Let $\mathcal{H}$ be an upper-semicontinuous hybrid system with compact state space. Then the return map r of $\mathcal{H}$ has a shift-invariant measure.*
- From an invariant measure for the return map, we can construct an invariant measure for the continuous dynamics.

Symbolic dynamics

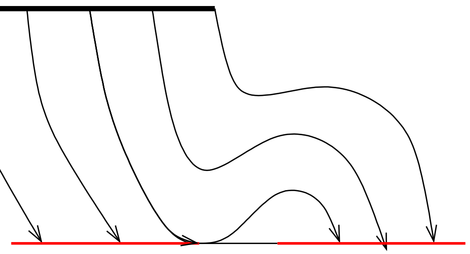
- Let $R_1, \ldots R_k$ be compact subsets of X (not necessarily disjoint or covering).
- A sequence (s_i) is a *itinerary* for a sequence (x_i) if $x_i \in R_{s_i}$ for all i. Note that a sequence (x_i) may have many itineraries, or none at all.
- If $f : X \to \mathcal{P}(X)$, then the *shift space* of f on $R_1, \ldots R_k$ is the set of all possible itineraries of orbits of f.
- If f is upper-semicontinuous, the shift space of f is compact.

Symbolic dynamics for hybrid systems

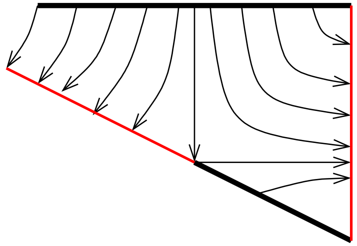
- Symbolic dynamics can be used to obtain an approximate description of the return map r of an upper-semicontinuous hybrid system.
- Since r is upper-semicontinuous, the shift space of r is compact.
- The sets R_k can often be chosed in a natural way depending on the reset map.
- Computing the shift space exactly is usually impossible, but some methods exist for computing lower bounds.

Bifurcation theory

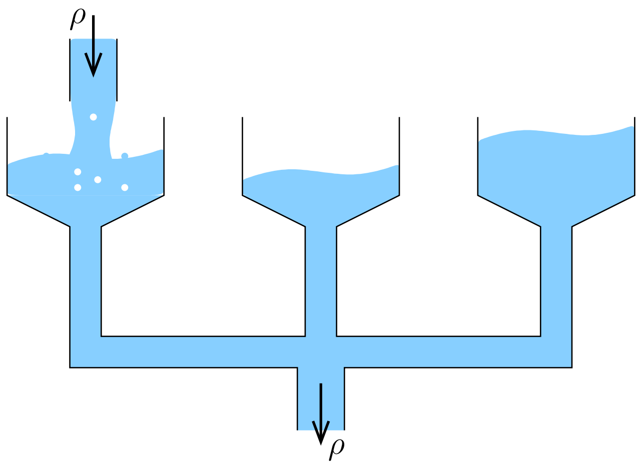
- Bifurcation theory for hybrid systems is not highly developed.
- Grazing bifurcations caused by a tangency with the guards are well-studied in the context of impact oscillators and give *period-adding* behaviour.



- Local bifurcations occur when a fixed point crosses a guard.



Example-filling tanks



- Tanks T_1, T_2 and T_3 with constant outflows ρ_1, ρ_2 and ρ_3.
- Single filler with inflow $\rho_{\text{in}} = \rho_{\text{out}} := \rho_1 + \rho_2 + \rho_3$.
- Volume of fluid in tank i is given by x_i.
- Total volume is preserved, so $x_1 + x_2 + x_3 = x_{\text{tot}}$.
- Switching law – as soon as a tank T_i empties, switch to filling T_i.

Example-filling tanks

- Zeno executions if two tanks empty at the same time.
- Invariant measure μ for return map with

$$\mu(\{x_1 = 0\}) = \frac{1}{2}\frac{\rho_1(\rho_2+\rho_3)}{\rho_1\rho_2+\rho_2\rho_3+\rho_3\rho_1}$$

- Symbolic dynamics of reset map — any sequence of switches possible.
- Bifurcation sequence stabilising a periodic orbit on altering switching behaviour.

Conclusions

- Upper-semicontinuity is a useful and natural property to impose on hybrid systems.
  - existence of limiting executions
  - natural treatment of Zeno executions
  - existence of shift invariant measures for the return map
  - simple construction of symbolic dynamics
- Hybrid systems can be expected to have a *much* richer bifurcation structure than continuous systems.

Further research

- Possible directions for further research
  - Use statistical methods to find optimal controls for chaotic systems.
  - Use symbolic dynamics with controls to stabilise some desired behaviour.
  - Use bifurcation theory to analyse and detect catastrophic system failures.
- Should be driven by practical problems.
- Very few examples in the literature — chaotic dynamics is hard to analyse and often unwanted anyway!