Synthesis of Networked Switching Linear Decentralized Controllers

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Outline

• Overview

• Decentralized controller synthesis

• Robust control over lossy networks

• Stochastic control over lossy networks

• Simulation results

• Conclusions and future work
Overview

•Spatially distributed system
  – linear model with coupled dynamics
  – sensors and actuators are distributed in space
  – distance limits transmission range

•Decentralized constrained control
  – each controller commands a subset of the available actuators
  – each controller exploits a subset of the available sensors
  – states and inputs are subject to constraints
Decentralized framework

Plant model
\[
x(t + 1) = Ax(t) + Bu(t)
\]

Constraints
\[
\|x(t)\|_2 \leq x_{\text{max}} \\
\|u(t)\|_2 \leq u_{\text{max}}
\]

How to account for network topology?

adjacency matrix
\[
\Lambda
\]

\[
\lambda_{ij} = \begin{cases} 
1 & \text{if sensor } s_j \text{ is linked to actuator } a_i, \\
0 & \text{otherwise.}
\end{cases}
\]

Linear decentralized regulator \( K \)

impose a structure on \( K \)

\[
\lambda_{ij} = 0 \Rightarrow k_{ij} = 0, \\
i = 1, \ldots, m, \\
j = 1, \ldots, n
\]
Decentralized linear controller synthesis

**Imposing stability:** Lyapunov function decrease at each time

\[ V_x(t + 1) - V_x(t) \leq -x(t)'Q_x x(t) - u(t)'Q_u u(t) \]

\[ Q_x = Q_x' \succ 0, \quad Q_u = Q_u' \succeq 0 \]

\[ V_x(t) \triangleq x(t)'Px(t) \]

\[ V_x(t) \geq J_\infty(t) \triangleq \sum_{i=0}^{\infty} (x(t + i)'Q_x x(t + i) + u(t + i)'Q_u u(t + i)) \]

**Goal:** find \( \gamma \) such that \( x(t)'Px(t) \leq \gamma, \ \forall t \in \mathbb{N}_0 \) for any \( x(0) \in X_0 \)

while fulfilling constraints

By substituting \( P = \gamma Q^{-1} \), the control law is defined as \( K = YQ^{-1} \)

We impose the following conditions to **guarantee the decentralization** of \( K \)

\[
\begin{align*}
(\lambda_{ij} = 0) & \Rightarrow y_{ij} = 0 \\
(\lambda_{ij} = 0) \land (\lambda_{ih} = 1) & \Rightarrow q_{hj} = 0, \ q_{jh} = 0
\end{align*}
\]

*Example on slide 7*
Decentralized linear controller synthesis

**Theorem:** Solving the following SDP for a given network topology $\Lambda$ we obtain a decentralized feedback control law which stabilizes the system and fulfills constraints.

\[
\begin{align*}
\min_{\gamma, Q, Y} \quad & \gamma \\
\text{s.t.} \quad & 
\begin{bmatrix}
Q & * & * & * \\
AQ + BY & Q & * & * \\
Q^{1/2} Q & 0 & \gamma I_n & * \\
Q^{1/2} Y & 0 & 0 & \gamma I_m
\end{bmatrix} \succeq 0, \\
& 
\begin{bmatrix}
Q & (AQ + BY)' \\
AQ + BY & x_{\max}^2 I_n
\end{bmatrix} \succeq 0, \\
& 
\begin{bmatrix}
u_{\max}^2 I_m & Y \\
Y' & Q
\end{bmatrix} \succeq 0, \\
& 
\begin{bmatrix}
1 & v_i' \\
v_i & Q
\end{bmatrix} \succeq 0, \quad i = 1, \ldots, n_v,
\end{align*}
\]

Lyapunov function decrease condition

State constraints satisfaction

Input constraints satisfaction

Initial condition $x(0) \in X_0$

\[v = \text{vertex}(X_0)\]

Decentralized structure

\[
(\lambda_{ij} = 0) \Rightarrow y_{ij} = 0 \\
(\lambda_{ij} = 0) \land (\lambda_{ih} = 1) \Rightarrow q_{hj} = 0, \quad q_{jh} = 0
\]

\[
i = 1, \ldots, m, \quad j = 1, \ldots, n, \quad h = 1, \ldots, n
\]
Decentralized controller structure: example

\[ \Lambda = \begin{bmatrix} 0 & 0 & 1 \ 1 & 1 & 0 \ 0 & 0 & 1 \end{bmatrix} \]

\[ Y = \begin{bmatrix} 0 & 0 & \ast & \ast & \ast & \ast \\ \ast & \ast & 0 & 0 & \ast & \ast \\ 0 & 0 & \ast & \ast & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \ast & \ast & 0 & 0 & 0 \ 0 & 0 & \ast & 0 & \ast \ 0 & 0 & \ast & \ast & 0 \ 0 & 0 & \ast & \ast & 0 \end{bmatrix} \]

\[ K = \begin{bmatrix} 0 & 0 & \ast & \ast & \ast \\ \ast & \ast & 0 & 0 & \ast \\ 0 & 0 & \ast & \ast & 0 \end{bmatrix} \]

Structure constraints on Q may lead to conservativeness.

Relaxing these constraints will allow for a more general Lyapunov matrix which permit a potentially better solution of the SDP.
Robust control over lossy networks

Transmissions over lossy links can be lost. A combinatorial number of possible scenarios have to be considered.

\[ L = \sum_{i=1}^{m} l_i \]

\[ \ell = 2^L \]

\[ \tilde{\Lambda}(t) \in \{\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_\ell\} \]

\[ \tilde{\Lambda}_h \in \{0, 1\}^{m \times n} \]

\[ \Lambda = \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \]
Robust analysis: switching controller

• State measurements available at current time instant vary in time

• Straightforward solutions
  1. Impose the less conservative structure on $K$, and replace lost measurements with estimations
  2. Impose the most conservative structure on $K$, and just discard measurements from lossy links

• Drawbacks
  1. Stability hard to guarantee in the general case
  2. High conservativeness: we do not exploit all information we have

Idea: design one controller for each possible network configuration

$$u(t) = \begin{cases} 
K_1x(t) & \text{if } \Lambda(t) = \Lambda_1, \\
K_2x(t) & \text{if } \tilde{\Lambda}(t) = \tilde{\Lambda}_2, \\
\vdots & \vdots \\
K_\ell x(t) & \text{if } \tilde{\Lambda}(t) = \tilde{\Lambda}_\ell.
\end{cases}$$

Problem: each controller needs to know the global network status in order to apply the correct control law!
Robust analysis: switching controllers

• Observations
  - Each input is computed via a matrix product which only involves the state measurements and the gain coefficients
  - Only the $i$-th row of the gains $K$ determines the command signal of the $i$-th controller
  - No communications the network configuration is not known locally

$$[\tilde{\Lambda}_h]_i = [\tilde{\Lambda}_j]_i \Rightarrow [K_h]_i = [K_j]_i$$  This removes ambiguity on the input to apply

- The actual number of control laws for the $i$-th actuator is $2^{l_i}$

$$u_i(t) = \begin{cases} 
F_1^i x(t) & \text{if } [\tilde{\Lambda}(t)]_i = \Gamma_1^i, \\
F_2^i x(t) & \text{if } [\tilde{\Lambda}(t)]_i = \Gamma_2^i, \\
\vdots & \vdots \\
F_{2^{l_i}}^i x(t) & \text{if } [\tilde{\Lambda}(t)]_i = \Gamma_{2^{l_i}}^i
\end{cases}$$

Note that $\Gamma_j^i$ are the possible configurations of measurements that the $i$-th actuator can receive. Thus, that is the only information it needs to compute its input.

- Sub-optimal, but does not assume global information on the network status
Switching robust controllers synthesis

**Theorem:**
Solving the following SDP for a given lossy network $\Lambda$, we obtain a set of decentralized feedback control laws which stabilize the plant and fulfill constraints for any combination of packet dropouts.

\[
\begin{align*}
\min_{\gamma, Q, \{Y\}} & \quad \gamma \\
\text{s.t.} & \quad 
\begin{bmatrix} Q & \ast & \ast & \ast \\
AQ + BY_h & Q & \ast & \ast \\
Q_\gamma^{1/2}Q & 0 & \gamma I_n & \ast \\
Q_u^{1/2}Y_h & 0 & 0 & \gamma I_m
\end{bmatrix} \succeq 0, \\
h = 1, \ldots, \ell, \\
\begin{bmatrix} Q & (AQ + BY_h)' \\
AQ + BY_h & x_\max^2 I_n
\end{bmatrix} \succeq 0, \\
h = 1, \ldots, \ell, \\
\begin{bmatrix} u_\max^2 I_m & Y_h \\
Y_h' & Q
\end{bmatrix} \succeq 0, \ h = 1, \ldots, \ell, \\
\begin{bmatrix} 1 & v_i' \\
v_i & Q
\end{bmatrix} \succeq 0, \ i = 1, \ldots, n_v, \\
\tilde{\lambda}_{ij}^h = 0 \Rightarrow y_{ij}^h = 0, \\
i = 1, \ldots, m, \ j = 1, \ldots, n, \ h = 1, \ldots, \ell, \\
(\tilde{\lambda}_{ij}^h = 0) \land (\tilde{\lambda}_{iw}^h = 1) \Rightarrow q_{wj} = 0, q_{jw} = 0, \\
i = 1, \ldots, m, \ j = 1, \ldots, n, \\
w = 1, \ldots, n, \ h = 1, \ldots, \ell, \\
[\Lambda_h]_i = [\tilde{A}_j]_i \Rightarrow [Y_h]_i = [Y_j]_i, \\
h, j = 1, \ldots, \ell, \ i = 1, \ldots, m
\end{align*}
\]

Linear controllers
\[K_w = Y_w Q^{-1}, \quad [Y_w Q^{-1}]_i = [Y_j Q^{-1}]_i, \ \forall i, j, w\]

Local controllers are not required to know the dropouts occurred in mother links.
Switching robust synthesis: LMI complexity

How many variables and constraints are involved?

Variables
Some $Y_h$ shares rows, hence some variables are shared

Upper bound of number of variables:
$$n \sum_{i=1}^{m} 2^{l_i} = nL$$

Constraints
no extra constraints needed to impose only local network knowledge

$$\min_{\gamma, Q, \{Y\}} \gamma$$
$$\begin{bmatrix} Q & * & * & * \\ AQ + BY_h & Q & * & * \\ Q_x^{1/2} & 0 & \gamma I_n & * \\ Q_u^{1/2} Y_h & 0 & 0 & \gamma I_m \end{bmatrix}_{h=1, \ldots, \ell} \succeq 0,$$
$$\begin{bmatrix} Q \quad (AQ + BY_h)' \quad x_{\max}^2 I_n \end{bmatrix}_{h=1, \ldots, \ell} \succeq 0,$$
$$\begin{bmatrix} u_{\max}^2 I_m & Y_h \\ Y_h' & Q \end{bmatrix} \succeq 0, \quad h = 1, \ldots, \ell,$$
$$\begin{bmatrix} 1 & v_i' \\ v_i & Q \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n_v,$$
$$\tilde{\lambda}_{ij}^h = 0 \Rightarrow y_{ij}^h = 0,$$
$$i = 1, \ldots, m, \quad j = 1, \ldots, n, \quad h = 1, \ldots, \ell,$$
$$\tilde{\lambda}_{ij}^h = 0 \land \tilde{\lambda}_{iw}^h = 1 \Rightarrow q_{wj} = 0, \quad q_{jw} = 0,$$
$$i = 1, \ldots, m, \quad j = 1, \ldots, n,$$
$$w = 1, \ldots, n, \quad h = 1, \ldots, \ell,$$
$$[\Lambda_h]_i = [\tilde{\Lambda}_j]_i \Rightarrow [Y_h]_i = [Y_j]_i,$$
$$h, j = 1, \ldots, \ell, \quad i = 1, \ldots, m.$$

How many variables and constraints are involved?
Stochastic network model

Stability in the mean-square sense

$$\lim_{t \to \infty} \mathbb{E} \left[ \|x(t)\|^2 \right] = 0.$$ 

The probability distribution of packet dropouts is modeled as a 2-state Markov chain

The diagram shows a 2-state Markov chain with states $Z_1$ and $Z_2$.

Transition matrix

$$T = \begin{bmatrix} q_1 & 1 - q_1 \\ 1 - q_2 & q_2 \end{bmatrix}$$

Emission matrix

$$E \in \mathbb{R}^{2 \times \ell} : e_{ij} = d_i \lambda_{s_0,j} (1 - d_i)^{s_1,j}, \quad i = 1, 2, \quad j = 1, \ldots, \ell.$$ 

For $s_{1,h} = \sum_{i=1, j=1}^{m, n} \lambda_{i,j}^h$, where $h = 1, \ldots, \ell$.

The number of -1 mapped in 1 into $\tilde{\Lambda}$.

For $s_{0,h} = \sum_{i=1, j=1}^{m, n} \left(1 - \lambda_{i,j}^h \right)$, where $h = 1, \ldots, \ell$.

The number of -1 mapped in 0 into $\tilde{\Lambda}$.

$\lambda_{i,j} = $ Probability of losing a packet while in $Z_i$.
A controller is synthesized for each configuration that may occur in each of the two Markov chain states

\[ u(t) = \begin{cases} 
K_{1,1} x(t) & \text{if } z(t) = Z_1, \tilde{\Lambda}(t) = \tilde{\Lambda}_1, \\
\vdots & \vdots \\
K_{1,\ell} x(t) & \text{if } z(t) = Z_1, \tilde{\Lambda}(t) = \tilde{\Lambda}_{\ell}, \\
K_{2,1} x(t) & \text{if } z(t) = Z_2, \tilde{\Lambda}(t) = \tilde{\Lambda}_1, \\
\vdots & \vdots \\
K_{2,\ell} x(t) & \text{if } z(t) = Z_2, \tilde{\Lambda}(t) = \tilde{\Lambda}_{\ell} 
\end{cases} \]

Switching stochastic Lyapunov function

\[ V_x(t) \triangleq \begin{cases} 
x(t)'P_1 x(t) & \text{if } z(t) = Z_1, \\
x(t)'P_2 x(t) & \text{if } z(t) = Z_2 
\end{cases} \]

we assume to know the Markov chain state at the current time step

Mean-square stability condition

\[ \mathbb{E} [V_x(t + 1 | t)] - V_x(t) \leq -x(t)'Q_x x(t) - \mathbb{E} [u(t | t)'Q_u u(t | t)] \]

next state? actual emission?

actual emission?
Switching robust controllers synthesis

\textbf{Theorem:}
Solving the following SDP for a given lossy network \( \Lambda \) with dropouts driven by a known Markov chain, we obtain a set of decentralized feedback control law which stabilize the plant in the mean-square sense

\[
P_j = \gamma Q_j^{-1}, \quad K_{j,h} = Y_{j,h} Q_j^{-1}, \quad \forall j, h
\]

\[
\begin{align*}
C_{j,1} &= \begin{bmatrix}
\sqrt{e_{j1}}t_{j1}(AQ_j + BY_{j,1}) \\
\cdots \\
\sqrt{e_{j\ell}}t_{j\ell}(AQ_j + BY_{j,1})
\end{bmatrix}, \\
C_{j,2} &= \begin{bmatrix}
\sqrt{e_{j1}}(Q_1^{1/2}Y_{j,1}) \\
\cdots \\
\sqrt{e_{j\ell}}(Q_1^{1/2}Y_{j,\ell})
\end{bmatrix}, \\
D_{j,1} &= \text{Blkdiag}\{Q_1, \ldots, Q_1, Q_2, \ldots, Q_2\}, \\
\ell \text{ times} &\quad \ell \text{ times}, \\
D_{j,2} &= \text{Blkdiag}\{\gamma I_m, \gamma I_m, \ldots, \gamma I_m\}, \\
\ell \text{ times}
\end{align*}
\]

\[
\begin{bmatrix}
Q_j & * & * & * \\
Q_j^{1/2}Q_j & \gamma I_n & * & * \\
C_{j,1} & 0 & D_{j,1} & * \\
C_{j,2} & 0 & 0 & D_{j,2}
\end{bmatrix} \preceq 0, \quad j = 1, 2,
\]

\[
\begin{bmatrix}
Q_j & (AQ_j + BY_{j,h})' \\
AQ_j + BY_{j,h} & x_{\text{max}}^2 I_n
\end{bmatrix} \succeq 0,
\]

\[
\begin{bmatrix}
u_{\text{max}} I_m & Y_{j,h} \\
Y_{j,h}' & Q_j
\end{bmatrix} \succeq 0,
\]

\[
\begin{bmatrix}
1 & v_i' \\
v_i & Q_j
\end{bmatrix} \succeq 0, \quad i = 1, \ldots, n_v, \quad j = 1, 2,
\]

\[
\lambda_{iw}^h = 0 \Rightarrow y_{iw}^j = 0, \quad j = 1, 2,
\]

\[
\begin{bmatrix}
\tilde{\lambda}_{iw}^h = 0
\end{bmatrix} \land \begin{bmatrix}
\tilde{\lambda}_{il}^h = 1
\end{bmatrix} \Rightarrow q_{lw}^j = 0, q_{wl}^j = 0,
\]

\[
\begin{bmatrix}
\tilde{\Lambda}_h \end{bmatrix} = \begin{bmatrix}
\tilde{\Lambda}_w \end{bmatrix}, \quad \forall j = 1, 2,
\]

\[
\begin{bmatrix}
[Y_{j,h}] & [Y_{j,w}]
\end{bmatrix} \succeq 0, \quad j = 1, 2,
\]

\[
\begin{bmatrix}
\gamma I_m, \gamma I_m, \ldots, \gamma I_m
\end{bmatrix}, \\
\ell \text{ times}
\]

\[
D_{j,2} = \text{Blkdiag}\{\gamma I_m, \gamma I_m, \ldots, \gamma I_m\}, \\
\ell \text{ times}
\]

\[
D_{j,2} = \text{Blkdiag}\{\gamma I_m, \gamma I_m, \ldots, \gamma I_m\}, \\
\ell \text{ times}
\]

\[
D_{j,2} = \text{Blkdiag}\{\gamma I_m, \gamma I_m, \ldots, \gamma I_m\}, \\
\ell \text{ times}
\]

\[
D_{j,2} = \text{Blkdiag}\{\gamma I_m, \gamma I_m, \ldots, \gamma I_m\}, \\
\ell \text{ times}
\]

\[
D_{j,2} = \text{Blkdiag}\{\gamma I_m, \gamma I_m, \ldots, \gamma I_m\}, \\
\ell \text{ times}
\]
Switching stochastic synthesis: LMI complexity

How many variables and constraints are involved?

Variables
With respect to the robust case the number of variables is doubled due to Markov states

Constraints
As the Markov chain has two states the matrix constraint is repeated. However its dimension has increased of $3 \cdot \ell$ due to the new terms

$$
\min_{\gamma, \{Q\}, \{Y\}} \gamma
$$

subject to

$$
\begin{bmatrix}
Q_j & \ast & \ast & \ast \\
Q_x^{1/2}Q_j & \gamma I_n & \ast & \ast \\
C_{j,1} & 0 & D_{j,1} & \ast \\
C_{j,2} & 0 & 0 & D_{j,2}
\end{bmatrix} \preceq 0, \quad j = 1, 2,
$$

$$
\begin{bmatrix}
Q_j & (AQ_j + BY_{j,h})' \\
AQ_j + BY_{j,h} & x_{\max}^2 I_n
\end{bmatrix} \succeq 0, \quad j = 1, 2, \quad h = 1, \ldots, \ell,
$$

$$
\begin{bmatrix}
u_{\max}^2 I_m & Y_{j,h} \\
Y_{j,h}' & Q_j
\end{bmatrix} \succeq 0, \quad j = 1, 2, \quad h = 1, \ldots, \ell,
$$

$$
\begin{bmatrix}
1 & v_i' \\
v_i & Q_j
\end{bmatrix} \succeq 0, \quad i = 1, \ldots, n_v, \quad j = 1, 2,
$$

$$
\tilde{\lambda}_{iw}^h = 0 \Rightarrow y_{iw}^{j,h} = 0, \quad j = 1, 2,
$$

$$
\tilde{\lambda}_{iw}^h = 0 \land \tilde{\lambda}_{il}^h = 1 \Rightarrow q_{lw}^j = 0, \quad q_{wl}^j = 0,
$$

$$
i = 1, \ldots, m, \quad j = 1, 2,
$$

$$\quad w, \ell = 1, \ldots, n, \quad h = 1, \ldots, n,
$$

$$\quad [\tilde{\Lambda}_h]_i = [\tilde{\Lambda}_w]_i \Rightarrow [Y_{j,h}]_i = [Y_{j,w}]_i, \quad j = 1, 2,
$$

$$\quad h, w = 1, \ldots, \ell, \quad i = 1, \ldots, m.$$
Randomly selected linear systems with fixed network topology

\[
\Lambda = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\
-1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

Simulation parameters

\[
N_{sim} = 50, \quad T_{sim} = 50
\]
\[
x_{\text{max}} = 25, \quad u_{\text{max}} = 3
\]
\[
Q_x = I_n, \quad Q_u = 10^{-2}I_m
\]
\[
x(0) \in X_0, \quad x_c = 7 \cdot 1_{8 \times 1}
\]
\[
X_0 = \{x_c\} + \{x \in \mathbb{R}^8 : \|x\|_\infty \leq 2\}
\]

Markov chain parameters

\[
q_1 = 0.8, \quad q_2 = 0.5, \quad d_1 = 0.1 \text{ and } d_2 = 0.5
\]

Network possible configurations

\[
\ell = 2^4 = 16
\]
Simulation example: results

(a) states

(b) inputs

---

robust

---

stochastic
Simulation example: results

Cumulated storage cost

\[ J_i = \sum_{t=1}^{T_{sim}} (\|Q_x x(t)\|_2 + \|Q_u u(t)\|_2) \]

<table>
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<th></th>
<th>( \mu(J_i) )</th>
<th>( \sigma(J_i) )</th>
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<td>Decentralized control</td>
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Conclusions and future works

• Conclusions
  – An SDP-based approach for decentralized controller synthesis has been proposed
  – Robust convergence with respect to random dropouts in the network is achieved assuming only local information on the network status
  – The approach is extended to the case where dropouts are described by random processes (Markov chain). Less conservative performances are obtained while fulfilling constraints, and stability in the mean-square sense is guaranteed

• Future works
  – Reduce the conservativeness of the proposed solutions by relaxing the structure constraints on the Lyapunov matrix (use parameter-dependent Lyap. functions)