Model Predictive Control of Hybrid Systems

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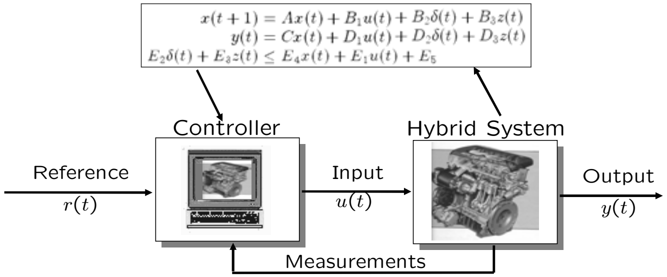


$$
\begin{aligned}
x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t)\\
y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t)\\
E_2\delta(t) + E_3z(t) &\le E_4x(t) + E_1u(t) + E_5
\end{aligned}
$$

- MODEL: a model of the plant is needed to predict the future behavior of the plant
- PREDICTIVE: optimization is based on the predicted future evolution of the plant
- CONTROL: control complex constrained multivariable plants

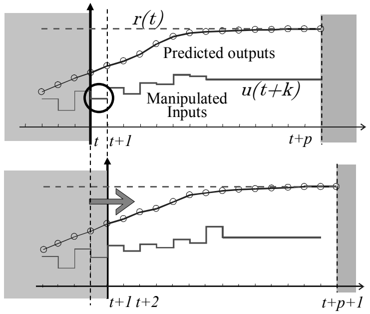
Receding Horizon Philosophy

- At time t :

  Solve an optimal control problem over a finite future horizon p :

  - minimize $|y-r|+\rho|u|$
  - subject to constraints

    $u_{min} \le u \le u_{max}$
    $y_{min} \le y \le y_{max}$

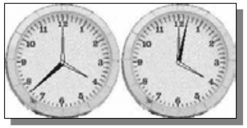


- Only apply the first optimal move $u^*(t)$
- Get new measurements, and repeat the optimization at time $t+1$

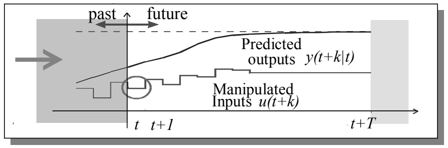
Advantage of on-line optimization: ***FEEDBACK!***

Receding Horizon - Example

MPC is like playing chess !



MPC for Hybrid Systems



Model Predictive (MPC) Control

- At time t solve with respect to $U \triangleq \{u(t), u(t+1), \ldots, u(t+T-1)\}$ the finite-horizon open-loop, optimal control problem:

$$\min_{u(t),\ldots,u(t+T-1)} \sum_{k=0}^{T-1} \|y(t+k|t) - r(t)\| + \rho\|u(t+k)\|$$
$$+\sigma(\|\delta(t+k) - \delta_r\| + \|z(t+k) - z_r\| + \|x(t+k|t) - x_r\|)$$

subject to MLD model

$$x(t|t) = x(t)$$
$$x(t+T|t) = x_r$$

- Apply only $u(t) = u^*(t)$ (discard the remaining optimal inputs)
- Repeat the whole optimization at time $t+1$

Closed-Loop Stability

Theorem 1 *Let $(x_r, u_r, \delta_r, z_r)$ be the equilibrium values corresponding to the set point r, and assume $x(0)$ is such that the MPC problem is feasible at time $t = 0$. Then $\forall Q, R \succ 0$, $\forall \sigma > 0$, the MPC controller stabilizes the MLD system*

$$\lim_{t\to\infty} y(t) = r$$
$$\lim_{t\to\infty} u(t) = u_r$$

$\lim_{t\to\infty} x(t) = x_r$, $\lim_{t\to\infty} \delta(t) = \delta_r$, $\lim_{t\to\infty} z(t) = z_r$, *and all constraints are fulfilled.*

(Bemporad, Morari 1999)

Proof: Easily follows from standard Lyapunov arguments

Stability Proof

- Assume we set the terminal constraint $x(t+T|t) = x_r$ in the optimal control problem
- Let $\mathcal{U}_t^*$ denote the optimal control sequence $\{u_t^*(0), \ldots, u_t^*(T-1)\}$
- Let $V(t) \triangleq J(\mathcal{U}_t^*, x(t))$=value function ⇒ Lyapunov function
- By construction, $\mathcal{U}_1 = \{u_t^*(1), \ldots, u_t^*(T-1), u_r\}$ is feasible @$t+1$
- Hence,

$$V(t+1) \boxed{\leq} J(\mathcal{U}_1, x(t+1)) = V(t) - \|y(t) - r\|_Q - \|u(t) - u_r\|_R -$$
$$\sigma(\|\delta(t) - \delta_r\| - \|z(t) - z_r\| - \|x(t) - x_r\|)$$

- Hence $V(t)$ is decreasing and lower-bounded by 0 $\Rightarrow \exists V_\infty = \lim_{t\to\infty} V(t)$ $\Rightarrow V(t+1) - V(t) \to 0$
- Hence, $\|y(t) - r\|_Q \to 0, \|u(t) - u_r\|_R \to 0, \ldots, \|x(t) - x_r\| \to 0$

Note: Global optimum not needed for convergence !

Hybrid MPC - Example

Switching System:

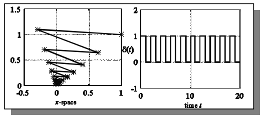
$$x(t+1) = 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1]\, x(t)$$

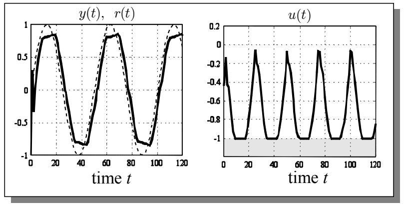
$$\alpha(t) = \begin{cases} \frac{\pi}{3} & \text{if } [1 \;\; 0]x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if } [1 \;\; 0]x(t) < 0 \end{cases}$$

Constraint: $-1 \leq u(t) \leq 1$

Open loop:



Closed loop:



Optimal Control of Hybrid Systems: Computational Aspects

MIQP Formulation of MPC

(Bemporad, Morari, 1999)

$$\min_\xi J(\xi, x(0)) = \sum_{t=0}^{T-1} y'(t)Qy(t) + u'(t)Ru(t)$$

$$\text{subject to } \begin{cases} x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\ y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\ E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5 \end{cases}$$

$$\xi = [u(0), \dots, u(T-1), \delta(0), \dots, \delta(T-1), z(0), \dots z(T-1)]'$$

$$\min_\xi \frac{1}{2}\xi'H\xi + x(0)'F\xi + \cancel{\frac{1}{2}x'(0)Yx(0)}$$

$$\text{subj. to } \quad G\xi \leq W + Sx(t)$$

Mixed Integer Quadratic Program (MIQP)

$u \in \mathbb{R}^{n_u}$, $\delta \in \{0,1\}^{n_\delta}$, $z \in \mathbb{R}^{n_z}$ $\Rightarrow$ $\xi \in \mathbb{R}^{(n_u+n_z)T} \times \{0,1\}^{n_\delta T}$

MILP Formulation of MPC

(Bemporad, Borrelli, Morari, 2000)

$$\min_\xi \quad J(\xi, x(0)) = \sum_{t=0}^{T-1} \|y(t)\|_\infty + \|Ru(t)\|_\infty$$

subject to MLD model

- Introduce slack variables: $\min |x|$ $\Rightarrow$ $\begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \epsilon \geq x \\ & \epsilon \geq -x \end{array}$

$$\begin{array}{lcl} \epsilon_k^x & \geq & \|Qy(t+k|t)\|_\infty \\ \epsilon_k^u & \geq & \|Ru(t+k)\|_\infty \end{array} \Rightarrow \begin{array}{lcll} \epsilon_k^x & \geq & [Qy(t+k|t)]_i & i=1,\dots,p, \; k=1,\dots,T-1 \\ \epsilon_k^x & \geq & -[Qy(t+k|t)]_i & i=1,\dots,p, \; k=1,\dots,T-1 \\ \epsilon_k^u & \geq & [Ru(t+k))]_i & i=1,\dots,m, \; k=0,\dots,T-1 \\ \epsilon_k^u & \geq & -[Ru(t+k))]_i & i=1,\dots,m, \; k=0,\dots,T-1 \end{array}$$

- Set $\xi \triangleq [\epsilon_1^x, \dots, \epsilon_{N_y}^x, \epsilon_1^u, \dots, \epsilon_{T-1}, U, \delta, z]$

Mixed Integer Linear Program (MILP)

$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u$$

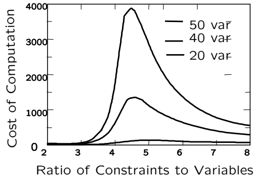
$$\text{s.t.} G\xi \leq W + Sx(t)$$

Mixed-Integer Program Solvers

- Mixed-Integer Programming is *NP*-hard

  Phase transitions have been found in computationally hard problems.

  BUT



(Monasson et al., *Nature*, 1999)

- General purpose Branch & Bound/Branch & Cut solvers available for MILP and MIQP (CPLEX, Xpress-MP, BARON, GLPK, ...)
  More solvers and benchmarks: `http://plato.la.asu.edu/bench.html`
- No need to reach global optimum (see proof of the theorem), although performance deteriorates

Good for large sampling times (e.g., 1 h) / expensive hardware ...

... but not for fast sampling (e.g. 10 ms) / cheap hardware !

Explicit Form of
Model Predictive Control

via Multiparametric Programming

On-Line vs. Off-Line Optimization

$$\min_U J(U,\boxed{x(t)}) \triangleq \sum_{k=0}^{T-1} \|Qy(t+k+1|t)\|_\infty + \|Ru(t+k)\|_\infty$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = \boxed{x(t)} \\ x(t+T|t) = 0 \end{cases}$$

- On-line optimization: given $x(t)$, solve the problem at each time step t

Mixed-Integer Linear Program (MILP)

- Off-line optimization: solve the MILP <u>for all</u> $x(t)$

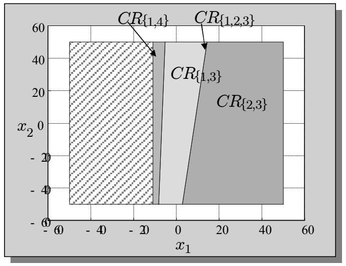
$$\begin{aligned} &\min_{\xi} J(\xi, \boxed{x(t)}) \triangleq f'\xi \\ &\text{s.t.} G\xi \le W + F\boxed{x(t)} \end{aligned}$$

multi-parametric Mixed Integer Linear Program (mp-MILP)

Example of Multiparametric Solution

Multiparametric LP ($\xi \in \mathbf{R}^2$)

$$\begin{aligned} &\min_{\xi} \quad -3\xi_1 - 8\xi_2 \\ &\text{s.t.} \begin{cases} \xi_1 + \xi_2 & \le & 13 + x_1 \\ 5\xi_1 - 4\xi_2 & \le & 20 \\ -8\xi_1 + 22\xi_2 & \le & 121 + x_2 \\ -4\xi_1 - \xi_2 & \le & -8 \\ -\xi_1 & \le & 0 \\ -\xi_2 & \le & 0 \end{cases} \end{aligned}$$



$$\xi(x) = \begin{cases} \begin{bmatrix} 0.00 & 0.05 \\ 0 & 0.06 \end{bmatrix} x + \begin{bmatrix} 11.85 \\ 9.80 \end{bmatrix} & \text{if} \begin{bmatrix} 0.02 & 0.00 \\ 0.00 & 0.02 \\ 0.00 & -0.02 \\ -0.12 & 0.01 \end{bmatrix} x \le \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ -1.00 \end{bmatrix} & \text{CR}_{\{2,3\}} \\ \begin{bmatrix} 0.73 & -0.03 \\ 0.27 & 0.03 \end{bmatrix} x + \begin{bmatrix} 5.50 \\ 7.50 \end{bmatrix} & \text{if} \begin{bmatrix} 0.00 & 0.02 \\ 0.00 & -0.02 \\ 0.12 & -0.01 \\ -0.15 & 0.00 \end{bmatrix} x \le \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} & \text{CR}_{\{1,3\}} \\ \begin{bmatrix} -0.33 & 0.00 \\ 1.33 & 0 \end{bmatrix} x + \begin{bmatrix} -1.67 \\ 14.67 \end{bmatrix} & \text{if} \begin{bmatrix} 0.00 & 0.02 \\ 0.00 & -0.02 \\ 0.15 & -0.00 \\ -0.09 & 0.00 \end{bmatrix} x \le \begin{bmatrix} 1.00 \\ 1.00 \\ -1.00 \\ 1.00 \end{bmatrix} & \text{CR}_{\{1,4\}} \end{cases}$$

Linear MPC

- Linear Model: $\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$ $\quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

- Constraints: $\begin{cases} u_{\text{min}} \le u(t) \le u_{\text{max}} \\ y_{\text{min}} \le y(t) \le y_{\text{max}} \end{cases}$

- Optimal control problem (quadratic performance index):

$$\min_{u_t,\ldots,u_{t+N-1}} \sum_{k=0}^{N-1} \left[x'_{t+k|t} Q x_{t+k|t} + u'_{t+k} R u_{t+k} \right] + x'_{t+N|t} P x_{t+N|t}$$

$$\begin{aligned} \text{s.t.} \quad & y_{min} \le y_{t+k|t} \le y_{max},\ k = 1, \ldots, N \\ & u_{min} \le u_{t+k} \le u_{max},\ k = 0, \ldots, N-1 \\ & x_{t|t} = x(t) \\ & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k},\ k = 0, \ldots, N-1 \\ & y_{t+k|t} = Cx_{t+k|t},\ k = 0, \ldots, N-1 \end{aligned}$$

$Q = Q' \succeq 0$, $R = R' \succ 0$, $P \succeq 0$, $(Q^{\frac{1}{2}}, A)$ detectable (e.g.: $Q = C'Q_yC$)

Linear MPC

- Substitution: $x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j}$

- Optimization problem:

$$\boxed{\begin{array}{rl} V(x(t)) = \frac{1}{2}x'(t)Yx(t) + \min\limits_{U} & \frac{1}{2}U'HU + x'(t)FU \\ \text{s.t.} & GU \leq W + Sx(t) \end{array}} \quad \begin{array}{l} \text{(quadratic)} \\ \text{(linear)} \end{array}$$

Convex QUADRATIC PROGRAM (QP)

- $U \triangleq [u'_t, \ldots, u'_{t+N-1}]' \in \mathbb{R}^s$, $s \triangleq Nm$, is the optimization vector
- $H = H' \succ 0$, and H, F, Y, G, W, S obtained from weights Q, R, P, and model matrices A, B, C

Multiparametric Quadratic Programming

(Bemporad et al., 2002)

$$\boxed{\begin{array}{ll} \min\limits_{U} & \frac{1}{2}U'HU + x'F'U + \cancel{\frac{1}{2}x'Yx} \\ \text{subj. to} & GU \leq W + Sx \end{array}}$$

$U \triangleq [u'_0 \ \ldots \ u'_{T-1}]'$

$U \in \mathbb{R}^r$, $r \triangleq n_u T$

$x \in \mathbb{R}^n$

- Objective: solve the QP **for all** $x \in X \subseteq \mathbb{R}^n$
- Assumption: $\begin{bmatrix} H & F \\ F' & Y \end{bmatrix} \succeq 0$ (always satisfied if QP problem originates from optimal control problem)

Linearity of the Solution

$x_0 \in X$ ⇒ solve QP to find $U^*(x_0)$, $\lambda^*(x_0)$

⇒ identify active constraints at $U^*(x_0)$

⇒ form matrices $\tilde{G}$, $\tilde{W}$, $\tilde{S}$ by collecting active constraints $\tilde{G}U^*(x_0) - \tilde{W} - \tilde{S}x_0 = 0$

KKT optimality conditions:

$$\boxed{\begin{array}{ll} (1)\ HU + Fx + G'\lambda = 0, & (2)\ \tilde{G}U - \tilde{W} - \tilde{S}x = 0 \\ (3)\ \lambda_i(G^iU - W^i - s^ix) = 0, & (4)\ \hat{G}U \leq \hat{W} + \hat{S}x \\ (5)\ \tilde{\lambda}_i \geq 0,\ \hat{\lambda}_i = 0 & \end{array}}$$

From (1) : $U = -H^{-1}(Fx + \tilde{G}'\tilde{\lambda})$

From (2) :

$$\boxed{\begin{array}{l} \tilde{\lambda}(x) = -(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W} + (\tilde{S} + \tilde{G}H^{-1}F)x). \\ U(x) = H^{-1}[\tilde{G}'(\tilde{G}H^{-1}\tilde{G}')^{-1}(\tilde{W}+(\tilde{S}+\tilde{G}H^{-1}F)x) - Fx] \end{array}}$$

➡ In some neighborhood of x_0, λ and U are explicit affine functions of x !

Determining a Critical Region

- Impose primal and dual feasibility:

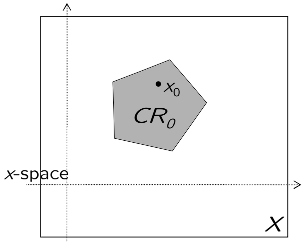
$$\boxed{\begin{array}{l} \hat{G}U(x) \leq \hat{W} + \hat{S}x \\ \tilde{\lambda}(x) \geq 0 \end{array}}$$

➡ linear inequalities in x !

- Remove redundant constraints

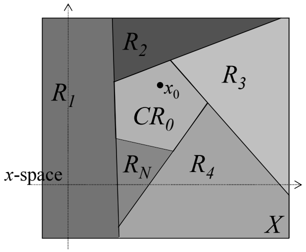
➡ critical region CR_0

$CR_0 = \{\mathcal{A}x \leq \mathcal{B}\}$



- CR_0 is the set of all and only parameters x for which $\tilde{G}, \tilde{W}, \tilde{S}$ is the optimal combination of active constraints at the optimizer

Multiparametric QP



$CR_0 = \{\mathcal{A}x \le \mathcal{B}\}$

$R_i = \{x \in X: \ \mathcal{A}^i x > \mathcal{B}^i, \ \mathcal{A}^j z \le \mathcal{B}^j, \ \forall j < i\}$

Theorem: $\{CR_0, R_1, \ldots, R_N\}$ is a partition of $X \subseteq \mathbf{R}^n$

Proceed iteratively: for each region R_i repeat the whole procedure with $X \leftarrow R_i$

The recursive algorithm terminates after a finite number of steps, because the number of combinations of active constraints is finite

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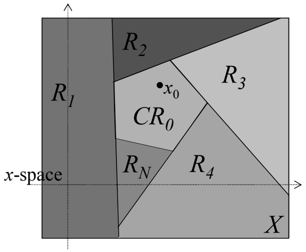
Multiparametric QP



$CR_0 = \{\mathcal{A}x \le \mathcal{B}\}$

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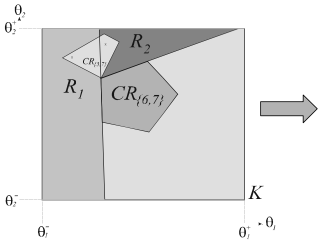
Theorem: $\{CR_0, R_1, \ldots, R_N\}$ is a partition of $X \subseteq \mathbf{R}^n$

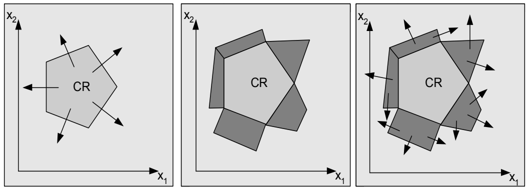
Note: while CR_0 is characterizing a set of active constraints, R_i is not



Keep track of the CR already explored, don't split CRs

Mp-QP – More efficient method

(Tøndel, Johansen, Bemporad, 2003)



The active set of a neighboring region is found by using the active set of the current region + knowledge of the type of hyperplane we are crossing:

$\widehat{G}^i U(x) \le \widehat{W}^i + \widehat{S}^i x \Rightarrow$ The corresponding constraint is **added** to the active set

$\tilde{\lambda}_j(x) \ge 0$ $\Rightarrow$ The corresponding constraint is **withdrawn** from the active set

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Mp-QP Properties

Theorem 1 *Consider a multi-parametric quadratic program with $H \succ 0$, $\left[\begin{smallmatrix} H & F \\ F' & Y \end{smallmatrix}\right] \succeq 0$. The set X^* of parameters x for which the problem is feasible is a polyhedral set, the value function $J^* : X^* \mapsto \mathbb{R}$ is piecewise quadratic, convex and continuous and the optimizer $U^* : X^* \mapsto \mathbb{R}^r$ is piecewise affine and continuous.*

$$U^*(x) = \arg\min_U \ \tfrac{1}{2}U'HU + x'F'U \quad \text{subj. to} \quad GU \le W + Sx$$

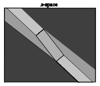
continuous, piecewise affine

$$V^*(x) = \tfrac{1}{2}x'Yx + \min_U \ \tfrac{1}{2}U'HU + x'F'U \quad \text{subj. to} \quad GU \le W + Sx$$

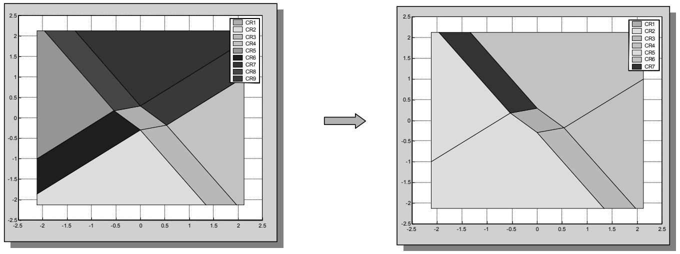
convex, continuous, piecewise quadratic, C^1 (if no degeneracy)

Corollary: The linear MPC controller is a continuous piecewise affine function of the state

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \le K_1 \\ \vdots & \quad \vdots \\ F_N x + G_N & \text{if} \quad H_N x \le K_N \end{cases}$$



Complexity Reduction



$$U(x) \triangleq [\boxed{u'_0(x)}\ u'_1(x)\ \dots\ u'_{N-1}(x)]'$$

Regions where the first component of the solution is the same can be joined (when their union is convex). (Bemporad, Fukuda, Torrisi, *Computational Geometry*, 2001)

Double Integrator Example

- System: $\boxed{y(t) = \frac{1}{s^2}\ u(t)}$ $\Longrightarrow$ $x(t+1) = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} x(t) + \begin{bmatrix}0\\ 1\end{bmatrix} u(t)$

  sampling + ZOH
  T_s=1 s
  $y(t) = [1\quad 0]\, x(t)$

- Constraints: $\boxed{-1 \le u(t) \le 1}$

- Control objective: minimize $\boxed{\sum_{t=0}^{\infty} y'(t)y(t) + \frac{1}{100}u^2(t)}$

  $u_{t+k} = K_{LQ}\ x(t+k|t)\ \ \forall k \ge N_u$

- Optimization problem: for *N_u=2*

$$H = \begin{bmatrix} 0.8365 & 0.3603\\ 0.3603 & 0.2059 \end{bmatrix},\ F = \begin{bmatrix} 0.4624 & 1.2852\\ 0.1682 & 0.5285 \end{bmatrix}$$

(cost function is normalized by max λ(H))

$$G = \begin{bmatrix} 1 & 0\\ -1 & 0\\ 0 & 1\\ 0 & -1 \end{bmatrix},\ W = \begin{bmatrix} 1\\ 1\\ 1\\ 1 \end{bmatrix},\ S = \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0 \end{bmatrix}$$

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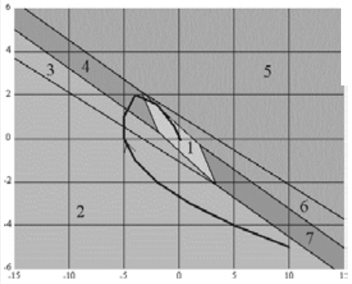
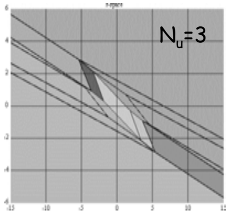
mp-QP solution

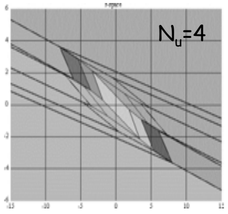
N_u=2

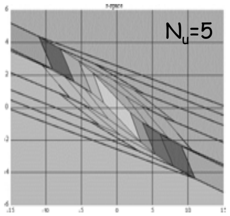
$$u(x) = \begin{cases} [\,0.8166\quad 1.7499\,]\,x & \text{if } \begin{bmatrix} 0.8166 & 1.7499\\ 0.8166 & 1.7499\\ 0.6124 & 0.4957\\ 0.6124 & 0.4957 \end{bmatrix} x < \begin{bmatrix} 1.0000\\ 1.0000\\ 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#1)}\\ 1.0000 & \text{if } \begin{bmatrix} 0.3864 & 1.0738\\ 0.2970 & 0.9333 \end{bmatrix} x \le \begin{bmatrix} 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#2)}\\ 1.0000 & \text{if } \begin{bmatrix} 0.9712 & 2.6991\\ 0.2970 & 0.9333\\ 0.8166 & 1.7499 \end{bmatrix} x \le \begin{bmatrix} 1.0000\\ 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#3)}\\ [\,0.5528\quad 1.5364\,]\,x + 0.4308 & \text{if } \begin{bmatrix} 0.9712 & 2.6991\\ 0.3864 & 1.0738\\ 0.6124 & 0.4957 \end{bmatrix} x < \begin{bmatrix} 1.0000\\ 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#4)}\\ -1.0000 & \text{if } \begin{bmatrix} 0.3864 & 1.0738\\ 0.2970 & 0.9333 \end{bmatrix} x \le \begin{bmatrix} 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#5)}\\ -1.0000 & \text{if } \begin{bmatrix} 0.9712 & 2.6991\\ 0.2970 & 0.9333\\ 0.8166 & 1.7499 \end{bmatrix} x < \begin{bmatrix} 1.0000\\ 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#6)}\\ [\,0.5528\quad 1.5364\,]\,x - 0.4308 & \text{if } \begin{bmatrix} 0.3864 & 1.0738\\ 0.9712 & 2.6991\\ 0.6124 & 0.4957 \end{bmatrix} x \le \begin{bmatrix} 1.0000\\ 1.0000\\ 1.0000 \end{bmatrix} & \text{(Region \#7)} \end{cases}$$

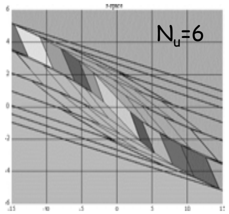


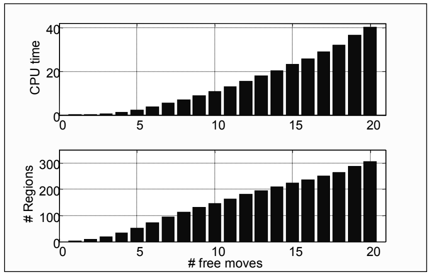
Complexity











Complexity

- Worst-case complexity analysis:

$M \triangleq \sum_{\ell=0}^{q} \binom{q}{\ell} = 2^q$ combinations of active constraints

$N_r \leq \sum_{k=0}^{M-1} k! q^k$ upper-bound to the number of regions

- Numerical Tests:

Free moves \ States	2	3	4	5
2	7	7	7	7
3	17	47	29	43
4	29	99	121	127

Number of regions in the state-space partition

Free moves \ States	2	3	4	5
2	3.02	4.12	5.05	5.33
3	10.44	26.75	31.7	70.19
4	25.27	60.20	53.93	58.61

Computation time (s)
[Matlab 5.3, Pentium II 300MHz]

Extensions

- Tracking of reference $r(t)$: $\delta u(t) = F(x(t), u(t-1), r(t))$

- Rejection of measured disturbance $v(t)$: $\delta u(t) = F(x(t), u(t-1), v(t))$

- Soft constraints: $u(t) = F(x(t))$

$y_{\min} - \epsilon \leq y(t+k|t) \leq y_{\max} + \epsilon$

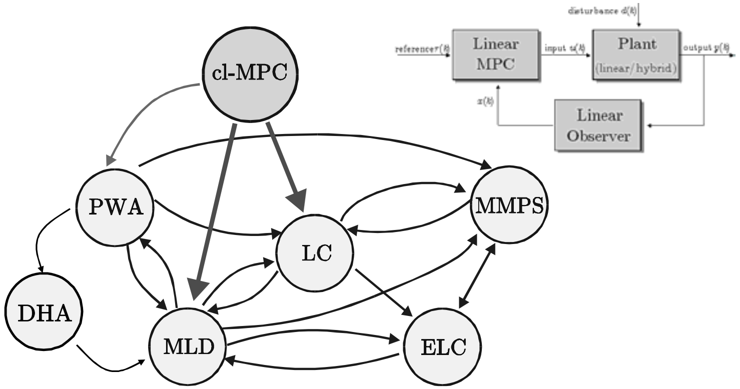
- Variable constraints: $u(t) = F(x(t), u_{\min}(t), \ldots, y_{\max}(t))$

$u_{\min}(t) \leq u(t+k) \leq u_{\max}(t)$
$y_{\min}(t) \leq y(t+k|t) \leq y_{\max}(t)$

- Linear norms: $\min_U J(U, x(t)) \triangleq \sum_{k=0}^{p} \|Qy(t+k|t)\|_\infty + \|Ru(t+k)\|_\infty$

(Bemporad, Borrelli, Morari, IEEE TAC, 2002)

Closed-Loop MPC and Hybrid Systems



Motivation:

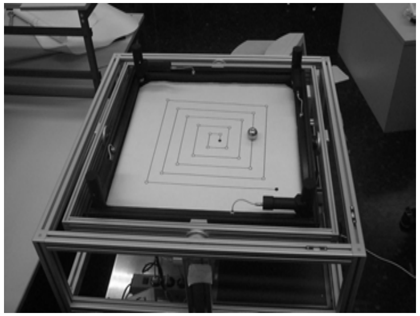
Use hybrid techniques to analyze closed-loop MPC systems !

(Bemporad, Heemels, De Schutter IEEE TAC, 2002)

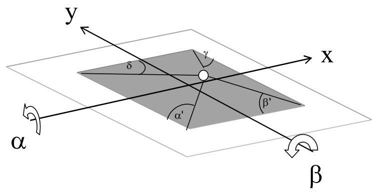
MPC Regulation of a Ball on a Plate

Task:

- Tune an MPC controller by simulation, using the *MPC Simulink Toolbox*
- Get the ***explicit solution*** of the MPC controller.
- Validate the controller on ***experiments***.



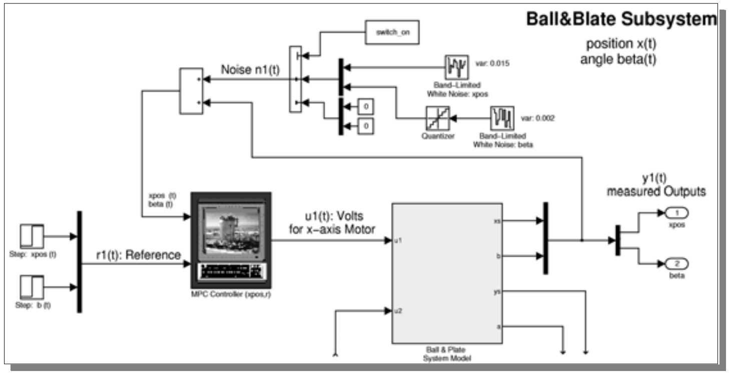
Ball&Plate Experiment



- Specifications:
  Angle: -17 deg ... +17deg
  Plate: -30 cm ...+30 cm
  Input Voltage: -10 V... +10 V
  Computer: PENTIUM166
  Sampling Time:30 ms
- Model: LTI 14 states
  Constraints on inputs and states

General Philosophy: (1) MPC Design

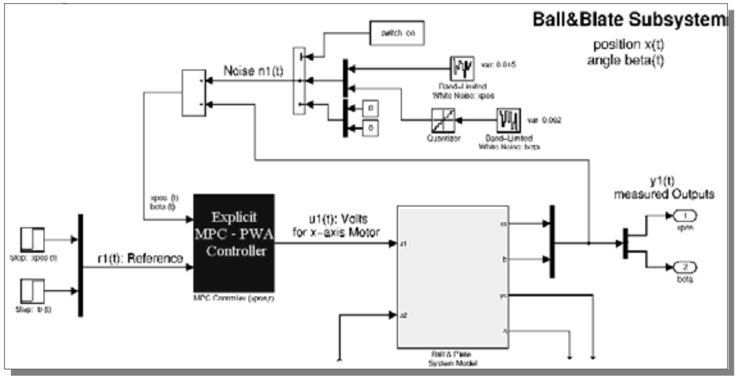
- Step 1: Tune the MPC controller (in simulation)



E.g: MPC Toolbox for Matlab
(Bemporad, Morari, Ricker, 2003)

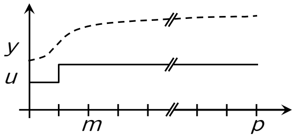
General Philosophy: (2) Implementation

- Step 2: Solve mp-QP and implement Explicit



E.g: Real-Time Workshop + xPC Toolbox

MPC Tuning



Sampling time: $T_s = 30\ ms$

Prediction horizon: $p = 50$

Free control moves: $m = 2$

Output constraint horizon: 1 *(soft constraint)*

Input constraint horizon: 1 *(hard constraint)*

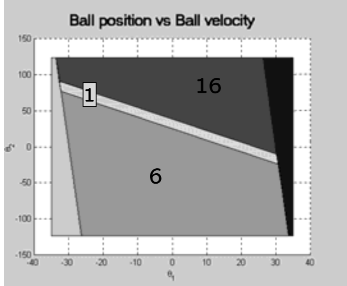
Weight on position error: *5*

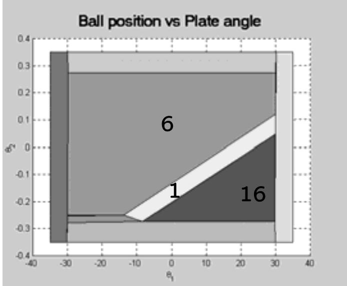
Weight on input voltage changes: *1*

Explicit MPC Solution

Controller: x: 22 Regions, y: 23 Regions

x-MPC: sections at $\dot{\alpha}_x=0$, $\alpha_x=0$, $u_x=0$, $r_x=18$, $r_\alpha=0$





Region 1: LQR Controller (near Equilibrium)

Region 6: Saturation at -10

Region 16: Saturation at +10

MPC Regulation of a Ball on a Plate

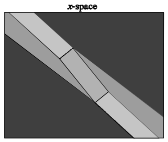
Design Steps:

- Tune an MPC controller by simulation, using the *MPC Simulink Toolbox* .
- Get the *explicit solution* of the MPC controller.
- ✓ Validate the controller on *experiments*.

Comments on Explicit MPC

- Multiparametric Quadratic Programs (mp-QP) can be solved efficiently
- Model Predictive Control (MPC) can be solved off-line via mp-QP
- Explicit solution of MPC controller $u = f(x)$ is Piecewise Affine

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \quad \vdots \\ F_N x + G_N & \text{if } H_N x \leq K_N \end{cases}$$



⇨ Eliminate heavy on-line computation for MPC

⇨ Make MPC suitable for fast/small/cheap processes

MILP Formulation of MPC

(Bemporad, Borrelli, Morari, 2000)

$$\min_\xi \quad J(\xi, x(0)) = \sum_{t=0}^{T-1} \|y(t)\|_\infty + \|Ru(t)\|_\infty$$

subject to MLD model

- Introduce slack variables: $\min |x| \Rightarrow \begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \epsilon \geq x \\ & \epsilon \geq -x \end{array}$

$$\begin{array}{lll} \epsilon_k^x & \geq & \|Qy(t+k|t)\|_\infty \\ \epsilon_k^u & \geq & \|Ru(t+k)\|_\infty \end{array} \Rightarrow \begin{array}{llll} \epsilon_k^x & \geq & [Qy(t+k|t)]_i & i=1,\ldots,p,\ k=1,\ldots,T-1 \\ \epsilon_k^x & \geq & -[Qy(t+k|t)]_i & i=1,\ldots,p,\ k=1,\ldots,T-1 \\ \epsilon_k^u & \geq & [Ru(t+k))]_i & i=1,\ldots,m,\ k=0,\ldots,T-1 \\ \epsilon_k^u & \geq & -[Ru(t+k))]_i & i=1,\ldots,m,\ k=0,\ldots,T-1 \end{array}$$

- Set $\xi \triangleq [\epsilon_1^x, \ldots, \epsilon_{N_y}^x, \epsilon_1^u, \ldots, \epsilon_{T-1}, U, \delta, z]$

$\Downarrow$

Mixed Integer Linear Program (MILP)

$$\min_\xi J(\xi, x(t)) = \sum_{k=0}^{T-1} \epsilon_i^x + \epsilon_i^u$$

$$\text{s.t.}\, G\xi \leq W + Sx(t)$$

Multiparametric MILP

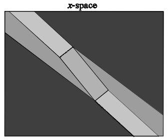
$$\begin{array}{ll} \min_{\xi=\{\xi_c,\xi_d\}} & f'\xi_c + d'\xi_d \\ \text{s.t.} & G\xi_c + E\xi_d \leq W + Fx \end{array} \qquad \begin{array}{l} \xi_c \in \mathbf{R}^n \\ \xi_d \in \{0,1\}^m \end{array}$$

- mp-MILP can be solved (by alternating MILPs and mp-LPs)

(Dua, Pistikopoulos, 1999)

- **Theorem:** The multiparametric solution $\xi^*(x)$ is piecewise affine
- **Corollary:** The MPC controller is piecewise affine in x

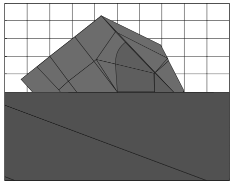
$$u(x) = \begin{cases} F_1 x + G_1 & \text{if } H_1 x \leq K_1 \\ \vdots & \vdots \\ F_N x + G_N & \text{if } H_N x \leq K_N \end{cases}$$



Solutions via Dynamic Programming

(Borrelli, Bemporad, Baotic, Morari, 2003)
(Mayne, ECC 2001)

- Explicit solutions to finite-time optimal control problems for PWA systems can be obtained using a combination of
  - Dynamic Programming
  - Multiparametric Linear (1-norm, ∞-norm), or or Quadratic (squared 2-norm) programming



Note: in the 2-norm case,
the partition may not be polyhedral

Hybrid Control Example (Revisited)

Hybrid Control - Example

Switching System:

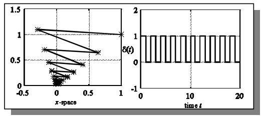
$$x(t+1) = 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1]\, x(t)$$

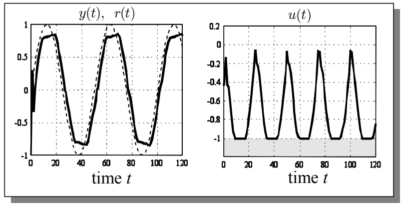
$$\alpha(t) = \begin{cases} \frac{\pi}{3} & \text{if } [1 \ \ 0]x(t) \geq 0 \\ -\frac{\pi}{3} & \text{if } [1 \ \ 0]x(t) < 0 \end{cases}$$

Constraint: $\boxed{-1 \leq u(t) \leq 1}$

Open loop:



Closed loop:



Hybrid MPC - Example

- MLD system

State x(t)	2 variables
Input u(t)	1 variables
Aux. binary vector δ(t)	1 variables
Aux. continuous vector z(t)	4 variables

- mp-MILP optimization problem

$$\min_{\{v_0^1\}} J(v_0^1, x(t)) \triangleq \sum_{k=0}^{1} \|Q_1(v(k)-u_e)\|_\infty + \|Q_2(\delta(k|t)-\delta_e)\|_\infty + \|Q_3(z(k|t)-z_e)\|_\infty + \|Q_4(x(k|t)-x_e)\|_\infty$$

subject to constraints

to be solved in the region $\boxed{\begin{array}{l} -5 \le x_1 \le 5 \\ -5 \le x_2 \le 5 \end{array}}$

- Computational complexity of mp-MILP

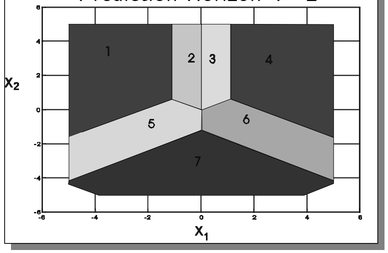
Linear constraints	84
Continuous variables	20
Binary variables	2
Parameters	2
Time to solve mp-MILP	3 min
Number of regions	7

---

mp-MILP Solution

$$u = \begin{cases} -1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.7296 \\ -1.0000 & 0.0000 \\ 0.0000 & 1.0000 \\ 94.5067 & -1.0000 \end{bmatrix} x \le \begin{bmatrix} -2.1680 \\ 5.0000 \\ 5.0000 \\ -105.1385 \end{bmatrix} \\ & \text{(Region \#1)} \\ \begin{bmatrix} 0.9238 & 0.0000 \end{bmatrix} x & \text{if } \begin{bmatrix} -188.1504 & 1.0000 \\ -20.3158 & -34.1997 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \le \begin{bmatrix} 204.3621 \\ 1.0000 \\ 0.0000 \\ 5.0000 \end{bmatrix} \\ & \text{(Region \#2)} \\ \begin{bmatrix} -0.9238 & -0.0000 \end{bmatrix} x & \text{if } \begin{bmatrix} -217.00 & 1.0000 \\ 38.1919 & -64.29 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} x \le \begin{bmatrix} 4.9708 \\ 1.0000 \\ 1.0000 \\ 5.0000 \end{bmatrix} \\ & \text{(Region \#3)} \\ -1.0000 & \text{if } \begin{bmatrix} -188.15 & -1.0000 \\ -1.0000 & -1.6858 \\ 1.0000 & 0.0000 \\ 1.0000 & 133.68 \end{bmatrix} x \le \begin{bmatrix} -204.39 \\ -2.1673 \\ 5.0000 \\ 665.60 \end{bmatrix} \\ & \text{(Region \#4)} \\ \begin{bmatrix} 0.4619 & -0.8000 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.0000 & -1.7193 \\ -1.0000 & 1.7295 \\ -1.0000 & 0.0000 \\ 13.4029 & 23.6383 \\ 43.4009 & 0.0000 \end{bmatrix} x \le \begin{bmatrix} 2.1387 \\ 2.1174 \\ 5.0000 \\ -1.0000 \\ -1.0000 \end{bmatrix} \\ & \text{(Region \#5)} \\ \begin{bmatrix} -0.4619 & -0.8000 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.0000 & 1.7175 \\ -20.3527 & 34.1997 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.7369 \\ 106.3247 & 1.0000 \end{bmatrix} x \le \begin{bmatrix} 2.0413 \\ -1.0000 \\ 0.0000 \\ 2.2345 \\ 525.0259 \end{bmatrix} \\ & \text{(Region \#6)} \\ 1.0000 & \text{if } \begin{bmatrix} 1.0000 & -1.6446 \\ -1.0000 & 1.7449 \\ -1.0000 & 0.0000 \\ -1.0000 & -1.6788 \\ 1.0000 & -263.137 \\ 1.0000 & 1.7369 \\ 1.0000 & 0.0000 \end{bmatrix} x \le \begin{bmatrix} 12.0940 \\ -2.2448 \\ 5.0000 \\ 12.3141 \\ 1329.55 \\ -2.2345 \\ 5.0000 \end{bmatrix} \\ & \text{(Region \#7)} \end{cases}$$

Prediction Horizon T=2

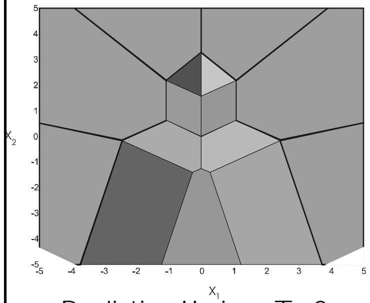


PWA law $\equiv$ MPC law

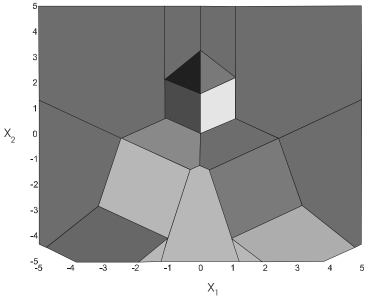
Linear constraints	84
Continuous variables	20
Binary variables	2
Parameters	2
Time to solve mp-MILP	3 min
Number of regions	7

---

mp-MILP Solution



Prediction Horizon T=3



Prediction Horizon T=4

Hybrid Control Example: Traction Control System

Vehicle Traction Control

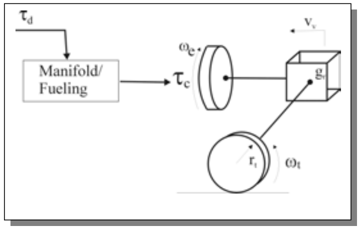
Improve driver's ability to control a vehicle
under adverse external conditions (wet or icy roads)

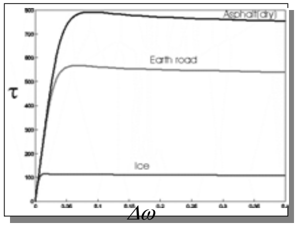
Model
nonlinear, uncertain,
constraints

Controller
suitable for real-time
implementation

MLD hybrid framework + optimization-based control strategy

Simple Traction Model





- Mechanical system

$$\dot\omega_e = \frac{1}{J_e}\left(\tau_c - b_e\omega_e - \frac{\tau_t}{g_r}\right)$$
$$\dot v_v = \frac{\tau_t}{m_v r_t}$$

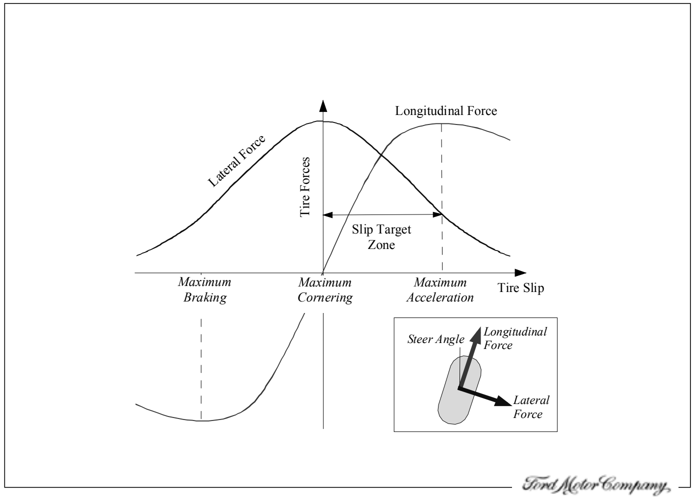
- Manifold/fueling dynamics

$$\tau_c = b_i \tau_d(t - \tau_f)$$

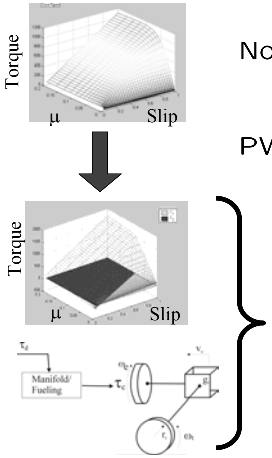
- Tire torque τ_t is a function of slip $\Delta\omega$ and road surface adhesion coefficient μ

$$\Delta\omega = \frac{\omega_e}{g_r} - \frac{v_v}{r_t}$$ Wheel slip

Tire Force Characteristics



Hybrid Model



Nonlinear tire torque $\tau_t = f(\Delta\omega,\ \mu)$

PWA Approximation
(PWL Toolbox, Julian, 1999)

HYSDEL
(Hybrid Systems Description Language)

→

Mixed-Logical
Dynamical (MLD)
Hybrid Model
(discrete time)

MLD Model

$$\begin{aligned} x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\ y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\ E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5 \end{aligned}$$

State x(t)	9 variables
Input u(t)	1 variable
Aux. Binary vars δ(t)	3 variables
Aux. Continuous vars z(t)	4 variables

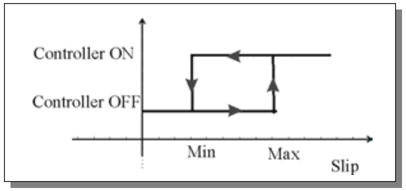
The MLD matrices are automatically generated in Matlab format by HYSDEL

Performance and Constraints

- Control objective:

$$\begin{array}{ll} \min \sum_{k=0}^{N} & \lvert \Delta\omega(k|t) - \Delta\omega_{des} \rvert \\ \text{subj. to.} & \text{MLD Dynamics} \end{array}$$

- Constraints:
  - Limits on the engine torque: $-20Nm \leq \tau_d \leq 176Nm$
- Logic Constraint:
  - Hysteresis



Experimental Apparatus

Experimental Apparatus

Experiment

- >500 regions
- 20ms sampling time
- Pentium 266Mhz + Labview

Ford Motor Company

Hybrid Control Example: Cruise Control System

Hybrid Control Problem

Renault Clio 1.9 DTI RXE

GOAL:

command gear ratio, gas pedal, and brakes to **track** a desired speed and minimize consumption

---

Hybrid Model

- Vehicle dynamics

$$\boxed{m\ddot{x} = F_e - F_b - \beta\dot{x}}$$

$\dot{x}$ = vehicle speed

F_e = traction force

F_b = brake force

⇒ discretized with sampling time $T_s = 0.5\ \mathrm{s}$

- Transmission kinematics

$$\boxed{\omega = \frac{R_g(i)}{k_s}\dot{x}}$$

$$\boxed{F_e = \frac{R_g(i)}{k_s}M}$$

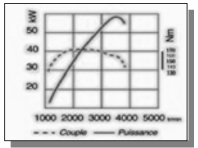
ω = engine speed

M = engine torque

i = gear

Hybrid Model

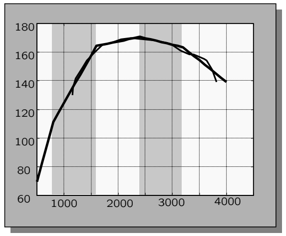
- Engine torque $\boxed{-C_e^-(\omega) \le M \le C_e^+(\omega)}$
- Max engine torque $\boxed{C_e^+(\omega)}$



http://www.renault.fr

Piecewise-linearization: $\Longrightarrow$

(PWL Toolbox, Juliãn, 1999)



requires: 4 binary aux variables
4 continuous aux variables

- Min engine torque $\boxed{C_e^-(\omega) = \alpha_1 + \beta_1\omega}$

Hybrid Model

- Gear selection: for each gear #i,

  define a binary input $\boxed{g_i \in \{0,1\}}$

- Gear selection (traction force):

  $\boxed{F_e = \frac{R_g(i)}{k_s}M}$ depends on gear #i

  define auxiliary continuous variables:

  $\boxed{\text{IF } \ g_i = 1 \ \text{ THEN } \ F_{ei} = \frac{R_g(i)}{k_s}M \ \text{ ELSE } \ 0}$

  $\Longrightarrow$ $\boxed{F_e = F_{eR} + F_{e1} + F_{e2} + F_{e3} + F_{e4} + F_{e5}}$

- Gear selection (engine/vehicle speed):

  $\boxed{\omega = \frac{R_g(i)}{k_s}\dot{x}}$ similarly, also requires 6 auxiliary continuous variables

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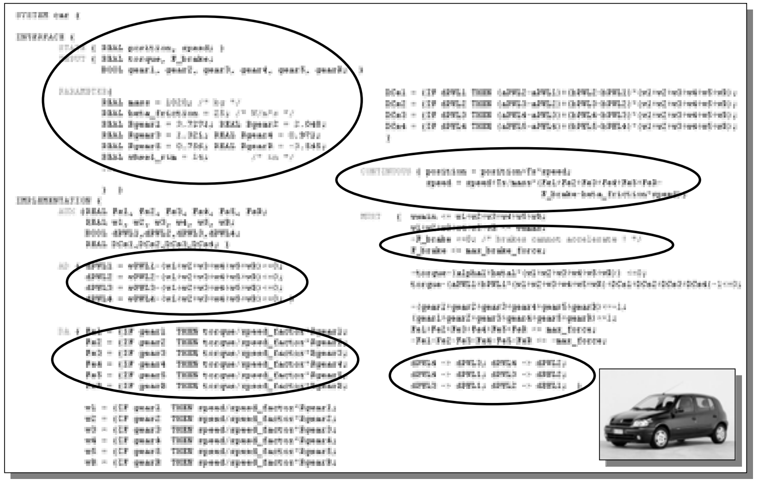
Hybrid Model

- MLD model

$$\boxed{\begin{aligned} x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) \\ y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \\ E_2\delta(t) + E_3z(t) &\le E_4x(t) + E_1u(t) + E_5 \end{aligned}}$$

- 2 continuous states: x, v (vehicle position and speed)
- 2 continuous inputs: M, F_b (engine torque, brake force)
- 6 binary inputs: $g_R, g_1, g_2, g_3, g_4, g_5$ (gears)
- 1 continuous output: v (vehicle speed)
- 16 auxiliary continuous vars: (6 traction force, 6 engine speed, 4 PWL max engine torque)
- 4 auxiliary binary vars: (PWL max engine torque breakpoints)
- 96 mixed-integer inequalities

Hysdel Model



http://control.ethz.ch/~hybrid/hysdel

Hybrid Controller

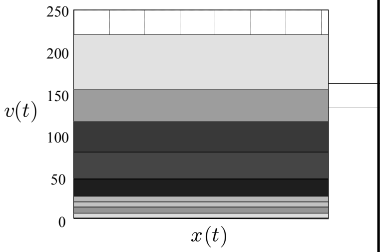
- Max-speed controller

$$\max_{u_t} J(u_t, x(t)) \triangleq v(t+1|t)$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

MILP optimization problem

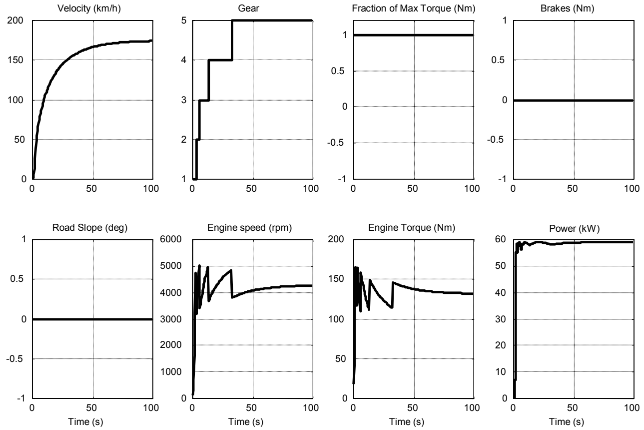
Linear constraints	96
Continuous variables	18
Binary variables	10
Parameters	1
Time to solve mp-MILP (Sun Ultra 10)	45 s
Number of regions	11



($x(t)$ is irrelevant)

Hybrid Controller

- Max-speed controller



Hybrid Controller

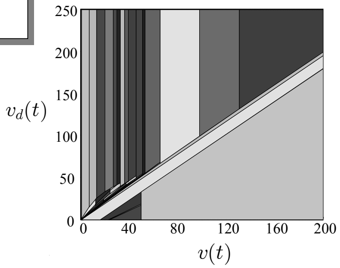
- Tracking controller

$$\min_{u_t} J(u_t, x(t)) \triangleq |v(t+1|t) - v_d(t)| + \rho|\omega|$$

$$\text{subj. to} \begin{cases} \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

MILP optimization problem

Linear constraints	98
Continuous variables	19
Binary variables	10
Parameters	2
Time to solve mp-MILP (Sun Ultra 10)	27 m
Number of regions	49

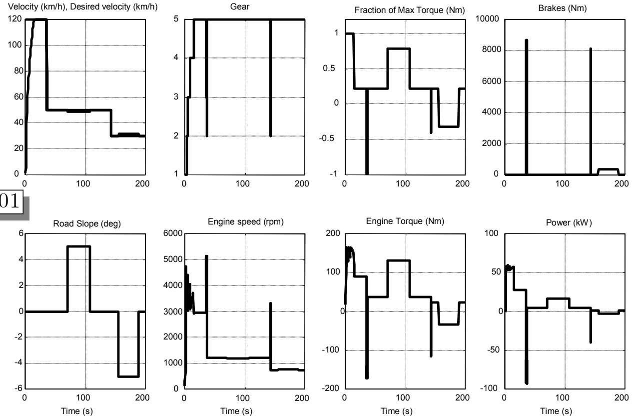


Hybrid Controller

- Tracking controller

$$\min_{u_t} J(u_t, x(t)) \triangleq |v(t+1|t) - v_d(t)| + \rho|\omega|$$

$\rho = 0.001$



Hybrid Controller

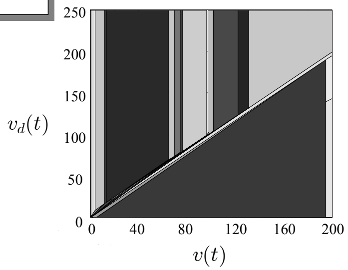
- Smoother tracking controller

$$\min_{u_t} J(u_t, x(t)) \triangleq |v(t+1|t) - v_d(t)| + \rho|\omega|$$

$$\text{subj. to} \begin{cases} |v(t+1|t) - v(t)| < T_s a_{\max} \\ \text{MLD model} \\ x(t|t) = x(t) \end{cases}$$

MILP optimization problem

Linear constraints	100
Continuous variables	19
Binary variables	10
Parameters	2
Time to solve mp-MILP (Sun Ultra 10)	28 m
Number of regions	**54**



Hybrid Controller

- Smoother tracking controller