Control Synthesis for Piecewise-Affine Hybrid Systems on Polytopes

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Outline

- Problems of control synthesis, realization, and computability.
- Piecewise-affine hybrid systems (PAHS).
- Control synthesis for PAHS.
  (1) Control to admissible exit facets.
  (2) Control to exit.
  (3) Stabilization to fixed point.
- Realization of piecewise-affine hybrid systems.
- Computability of hybrid systems.
- Concluding remarks.

CWI - Approaches to control, realization, and computability of HS

1. **Control synthesis:** Computable sufficient conditions for existence of control laws and algorithms for control laws.
2. **Realization:** Characterization of sets of input-output trajectories which are representable as those of a hybrid system. Reachability and observability. (In general undecidable.)
3. **Computability:** For which subclass of nonlinear (hybrid) system is the reachable subset numerically approximable?

Remarks on control of hybrid systems

- Complexity of control synthesis is the main issue for control of hybrid systems.
- Theory of computation and complexity for discrete sets.
  (Concept of Turing machine. Decidable and undecidable problems.)
- For real numbers, complexity theory available in books:
  (1) Blum-Cucker-Shub-Smale.
  (2) K. Weihrauch (combination of analysis and computation).
  Needed are more concepts, theorems, and experience.
- Problems of reachability and of observability of PAHS are undecidable.
  (E.D. Sontag (1995); P.J. Collins, JHvS (CDC.2004)).

Def. Piecewise-affine hybrid system (PAHS, CT, Time-invariant)

$$
\begin{array}{rcl}
Q & & \text{finite state set, } U \subset \mathbb{R}^m,\ Y \subset \mathbb{R}^p,\ \text{polyhedral sets,} \\
X_q & \subset & \mathbb{R}^{n_q},\ \forall q \in Q,\ \text{closed polyhedral sets,} \\
\dot{x}_q(t) & = & A(q)x_q(t) + B(q)u(t) + a(q),\ x_q(t_o) = x_q^+, \\
y(t) & = & C(q)x_q(t) + D(q)u(t) + c(q), \\
e & \in & E_{cd},\ \text{if } x(t_1) \in G_q(e) \subset \partial X_q,\ \text{guard;} \\
 & & \text{event generated by continuous dynamics; then transition,} \\
q^+ & = & f(q^-, x_{q^-}^-, e),\ q_0, \\
x_{q^+}^+ & = & A_r(q^-, e, q^+)x_{q^-}^- + b_r(q^-, e, q^+).
\end{array}
$$

Assumptions: (1) Finite number of events at any time.

(2) Finite number of events on any finite interval (non-Zenoness).

Def. Affine system on a polytope. (**FDAPS**)

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + a, \;\; x(t_0) = x_0 \in X_0 \subseteq X, \\
y(t) &= Cx(t) + Du(t) + c, \\
& \quad U \subset \mathbb{R}^m, \;\; Y \subset \mathbb{R}^p, \;\; \text{polytopes}, \\
& \quad X \subset \mathbb{R}^n, \;\; \text{closed full-dim. polytope}, \\
t_1 &= \inf \left\{ \begin{array}{l} t \in T \cup \{+\infty\} \mid x(t) \in F_{n-1,r} \subset \partial X \\ \text{and state attempts to exit from polytope} \end{array} \right\}, \\
F_{n-1,r} & \quad \text{called } \textbf{exit facet;} \\
& \quad \textbf{lifetime} \text{ of state trajectory:} \\
T_1 &= [t_0, \infty), \;\; \text{if } t_1 = \infty, \;\; \text{or}, \\
T_1 &= [t_0, t_1] \subset R_+, \;\; \text{if } t_1 < \infty, \\
& \quad \text{then } u, x, y \text{ defined on } T_1.
\end{aligned}
$$

Concepts - Polytopes

Def. A **polytope** $P \subset \mathbb{R}^n$ is defined to be
a finite intersection of closed half spaces which, moreover, is bounded,

$$\begin{aligned} P &= \cap_{i=1}^{m}\{x \in \mathbb{R}^n | n_i^T x \leq q_i\}, \\ \dim(P) &= \dim(\mathrm{affh}(P)). \end{aligned}$$

(Equivalently, a polytope is the convex hull of a finite number of vectors.)
Representations,

$$\begin{aligned} P &= \{x \in \mathbb{R}^n | N^T x \leq q\}, \text{ **implicit form;**} \\ &= \{x \in \mathbb{R}^n | \exists y \in S_+^k,\ x = Ay\}, \text{ **explicit form.**} \end{aligned}$$

Def. A **simplex** is a polytope for which there exists $r \in Z_+$,

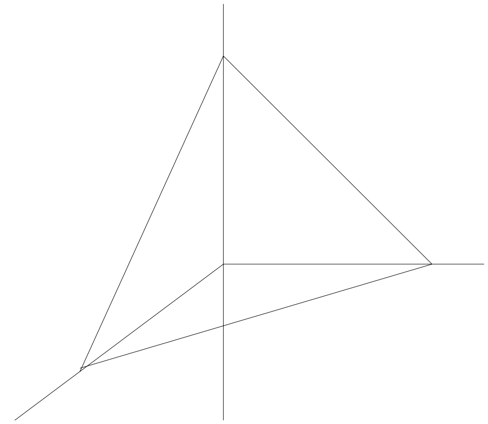
$$\begin{aligned} P &= \mathrm{convh}(\{v_1, \ldots, v_{r+1}\}) \subset \mathbb{R}^n, \\ \dim(P) &= r. \end{aligned}$$

Full-dimensional simplex if $\dim(P) = n$.

Concepts - Faces and facets of polytopes

Def. Consider polytope

$$P = \{x \in \mathbb{R}^n | N^T x \leq q\}.$$



Face of P defined as the set,

$$\begin{aligned} F &= P \cap \{x \in \mathbb{R}^n | N_s^T x = q_s\}, \\ \dim(F) &= \dim(\text{affh}(F)). \end{aligned}$$

Facet of P is face F such that,

$$\dim(F) = \dim(P) - 1.$$

Notation

$$\begin{aligned} \mathbf{F_{n_P-1}}(P) &= \{F_{n_P-1,i} \subset P | \forall\ i \in Z_r\}, \text{ set of facets,} \\ F_{n_P-1,i} &= P \cap \{x \in \mathbb{R}^n | n_i^T x = q_i\}, \text{ facet } i \in \mathbb{Z}_r = \{1, 2, \ldots r\}. \end{aligned}$$

Facet is intersection of polyhedral set with a supporting hyperplane.

Lattice of faces fully describes combinatorial structure of a polyhedral set.

CWI approach to control synthesis for PAHS

1. Decomposition into discrete and continuous dynamics.
2. Control of affine systems on polytopes.
   (a) Control-to-exit.
   (b) Stabilization-to-fixed-point.

   This is a form of **geometric control**.
3. Control at discrete level: reachability and supervisory control.

Remarks Alternative approaches:

- PAHS-CT UCB game theory approach.
- PAHS-DT ETHZ Predictive control and computational approach.

Problem Control synthesis for PAHS on a simplex

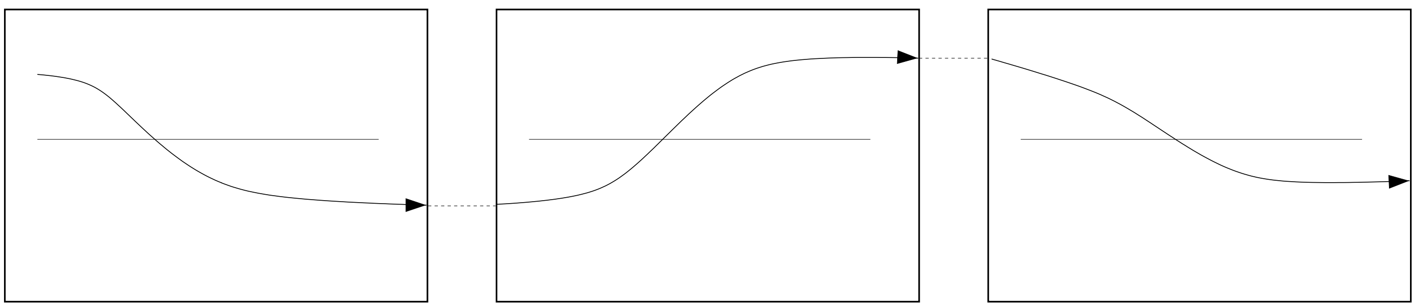
$$\begin{aligned}\dot{x}_q(t) &= A(q)x_q(t) + B(q)u(t) + a(q),\ x_q(t_0) = x_{q,0},\\ &\quad Q \text{ finite set, } U \subset \mathbb{R}^m \text{ polytope,}\\ &\quad X_q \subset \mathbb{R}^{n_q} \text{ simplex, } G_q(e) \subset \partial X_q \text{ guards contained in facets, } \forall q \in Q,\\ &\quad Q_u \subset Q \textbf{ unsafe locations,}\\ &\quad Q_s \subset Q \backslash Q_u \textbf{ start locations, } q_t \in Q \backslash Q_u \textbf{ target location.}\end{aligned}$$

Determine control laws,

$$\begin{aligned}k_q(x) &= F_q x + g_q,\ k_q : X_q \to U,\ \forall q \in Q,\\ &\quad \text{such that } \exists t_1 \in [t_0, \infty) \text{ and closed-loop system is such that}\\ &\quad (t_0, q_s, x_{q_s,s}) \in T \times Q_s \times X_{q_s} \ \mapsto\ (t_1, q_t, x_{q_t,t}) \in T \times Q \times X_{q_t},\\ &\quad \text{either stay at target location or converge to fixed point } x_{q_t,t} \in X_{q_t}.\end{aligned}$$

Remarks (1) Sufficient condition. (2) Computationally tractable.

Figure illustrating control synthesis for PAHS.



Problem Control-to-admissible-exit-facets

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0,\\ &\quad X \subset \mathbb{R}^n \text{ simplex},\ U \subset \mathbb{R}^m \text{ polytope},\\ &\quad I \subseteq \mathbb{Z}_{N+1} = \{1, \ldots, N+1\},\\ &\quad \{F_i \in \mathbf{F_{N+1}}(X), i \in I\} \text{ admissible exit facets.}\end{aligned}$$

Determine an affine control law,

$$\begin{aligned}k(x) &= Fx + g,\\ \dot{x}(t) &= (A + BF)x(t) + (a + Bg),\ x(t_0) = x_0, \quad \text{closed-loop system},\end{aligned}$$

such that,
if the state trajectory of the closed-loop system leaves the simplex
then it does so through one of the admissible exit facets.

Theorem Control-to-admissible-exit-facets

Consider an affine system on a simplex.

There exists an affine control law for this problem

if and only if

$\exists\ u_1, \ldots, u_{N+1} \in U$ such that,

$n_i^T(Av_j + Bu_j + a) \leq 0, \ \ \forall i \in \mathbb{Z}_{N+1}\backslash I, \ \forall j \in \mathbb{Z}_{N+1}\backslash\{i\}.$

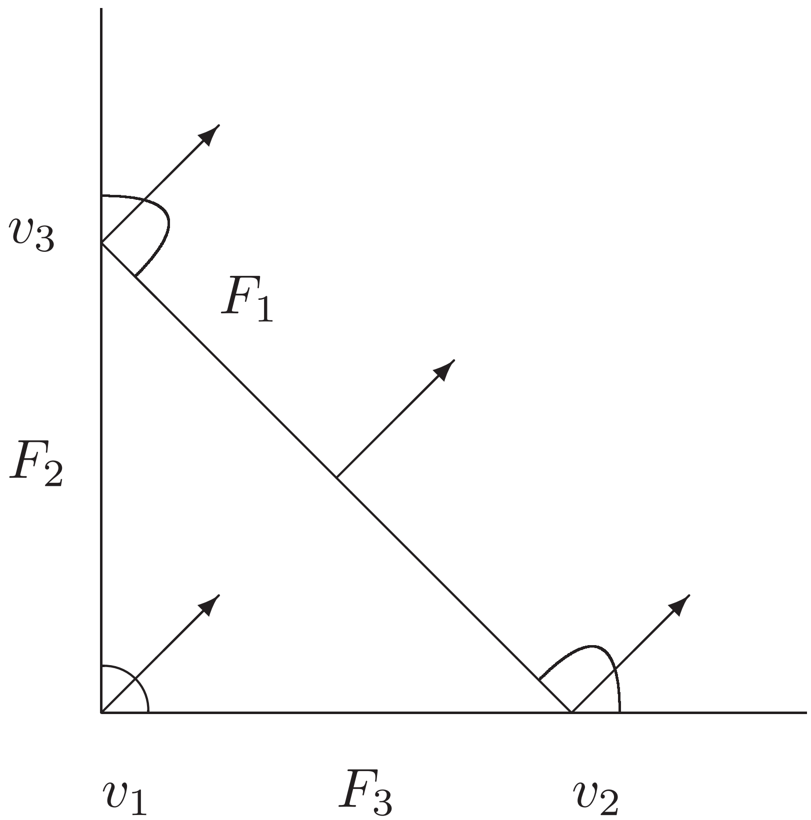
Then (F, g) are the unique solution of the equation,

$$\begin{pmatrix} v_1^T & 1 \\ \vdots & \vdots \\ v_m^T & 1 \end{pmatrix} \begin{pmatrix} F^T \\ g^T \end{pmatrix} = \begin{pmatrix} u_1^T \\ \vdots \\ u_{N+1}^T \end{pmatrix},$$

$k(x) = Fx + g,$ control law.

Remarks Linear inequalities solvable by computer programs.

Control to admissible exit facets



Problem **Control-to-exit**. Consider,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + a,\ x(t_0) = x_0, \\ & X \subset \mathbb{R}^N \text{ simplex, } I \subset \mathbb{Z}_{N+1}, \\ & \{F_i \in \mathbf{F_{N+1}}(X), i \in I\} \text{ admissible exit facets.} \end{aligned}$$

Determine an affine control law

$$k(x) = Fx + g,$$

such that the state trajectory of the closed-loop system leaves the polytope X via an admissible exit facet in finite time.

Theorem Existence of fixed points

Consider an autonomous affine system on a polytope

$$\dot{x}(t) = Ax(t) + a,\ x(t_0) = x_0,\ X \subset \mathbb{R}^N.$$

There exists a **fixed point**,

$$0 = Ax_f + a,\ \ x_f \in X,$$

if and only if

$$\exists x_0 \in X \text{ such that } \forall t \in [t_0, \infty),\ x(t, x_0) \in X.$$

Remark Condition checkable in terms of linear inequalities.

Theorem Control-to-exit

Consider an affine system on a simplex.

$$
\begin{aligned}
U_j &= \left\{ u \in U | n_i^T (A v_j + B u + a) \leq 0, \ \forall i \in \mathbb{Z}_{N+1} \backslash (I \cup \{j\}) \right\}, \\
W_j &= \text{vertices of } U_j, \ \ \forall j \in \mathbb{Z}_{N+1}.
\end{aligned}
$$

The problem is solvable if and only if

$$
\begin{aligned}
& \forall j \in \mathbb{Z}_{N+1} \ \exists w_j \in W_j \text{ such that}, \\
0 &\notin \operatorname{convh}(\{A v_j + B w_j + a | \forall j \in \mathbb{Z}_{N+1}\}).
\end{aligned}
$$

Remark

(1) Linear equalities to be checked for condition.

(2) Controllability condition: $U_j \neq \emptyset$, $\forall j \in \mathbb{Z}_{N+1}$.

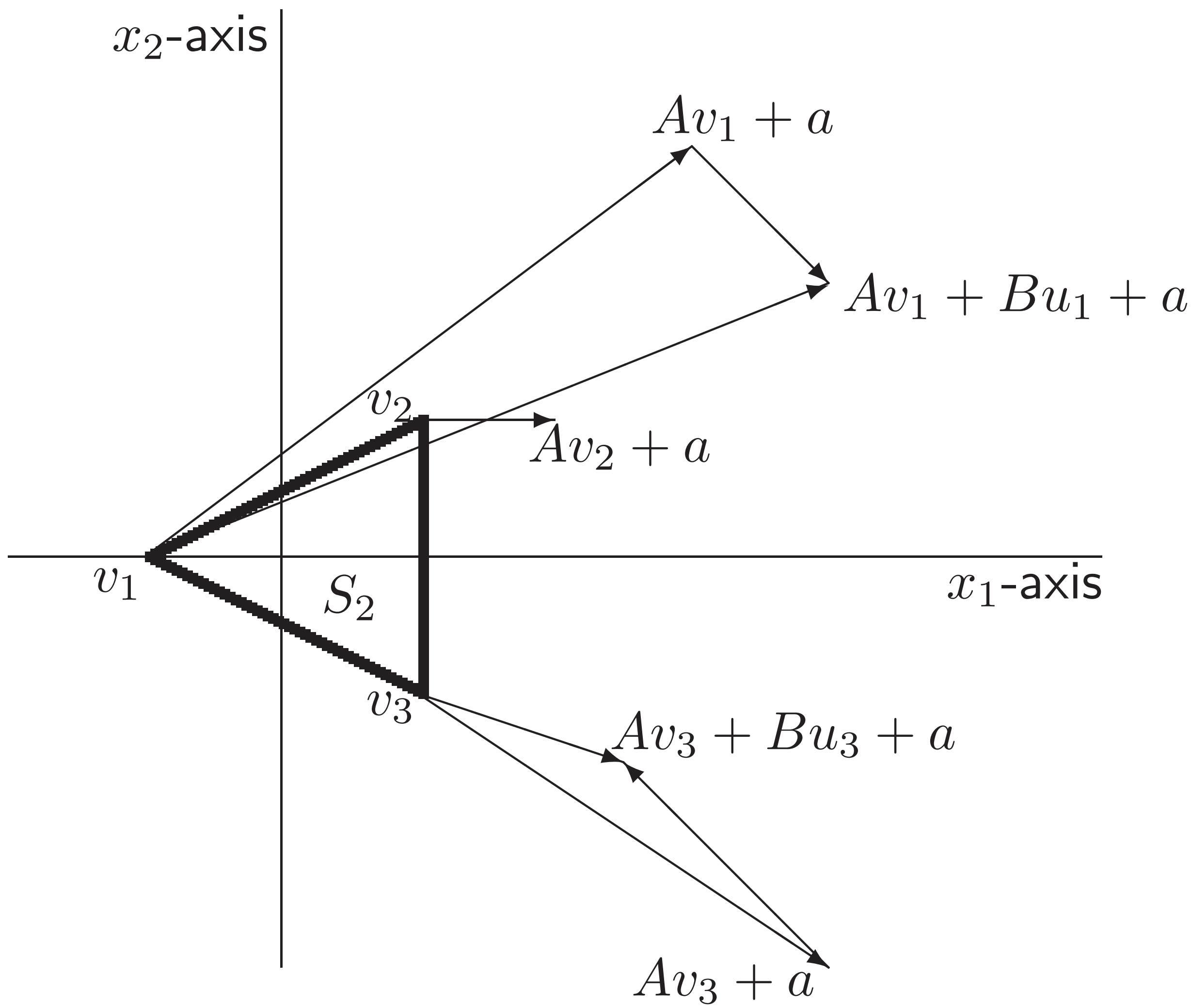
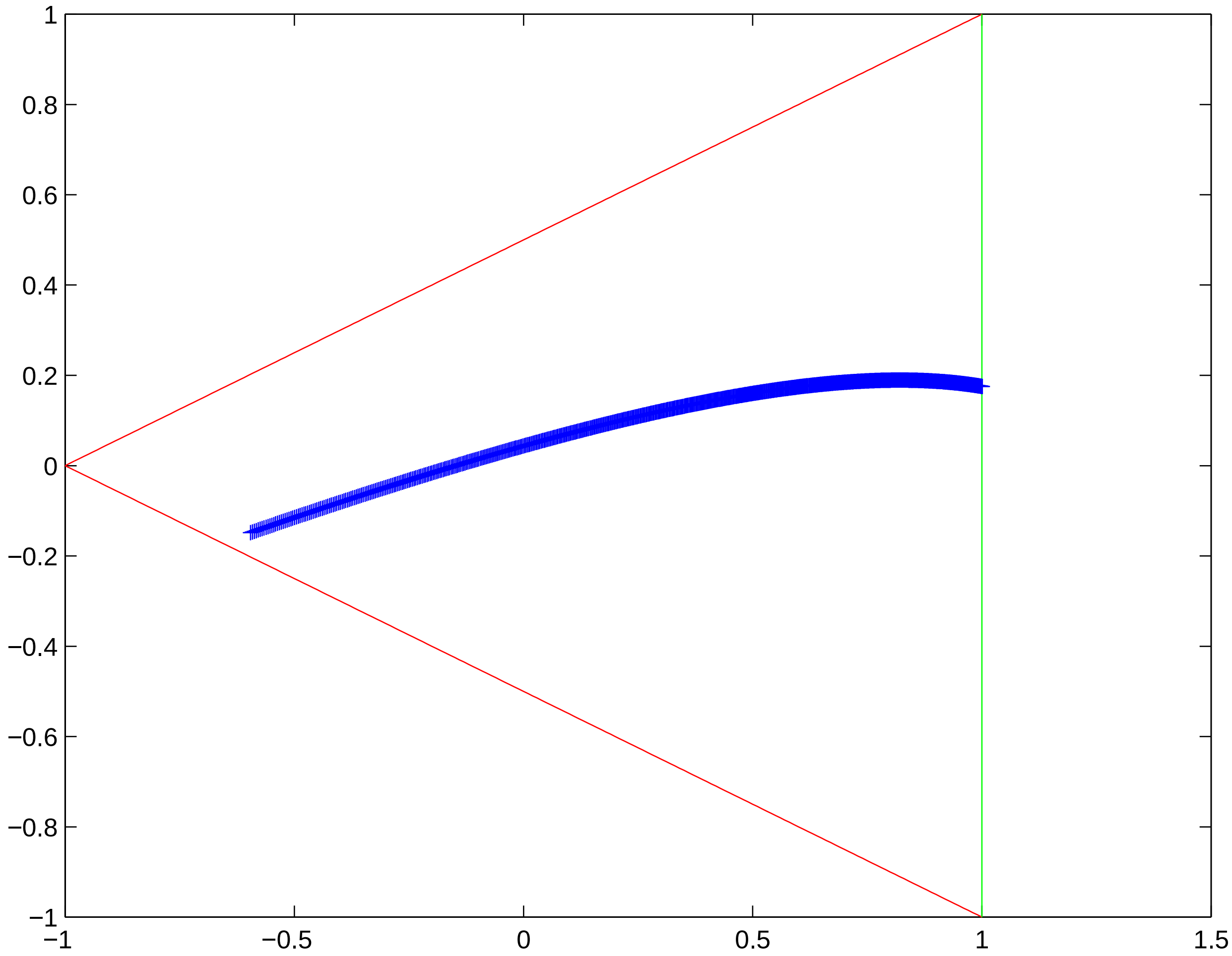


Figure 1: Control of the vector field $\dot{x}$ at the vertices of S_2


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
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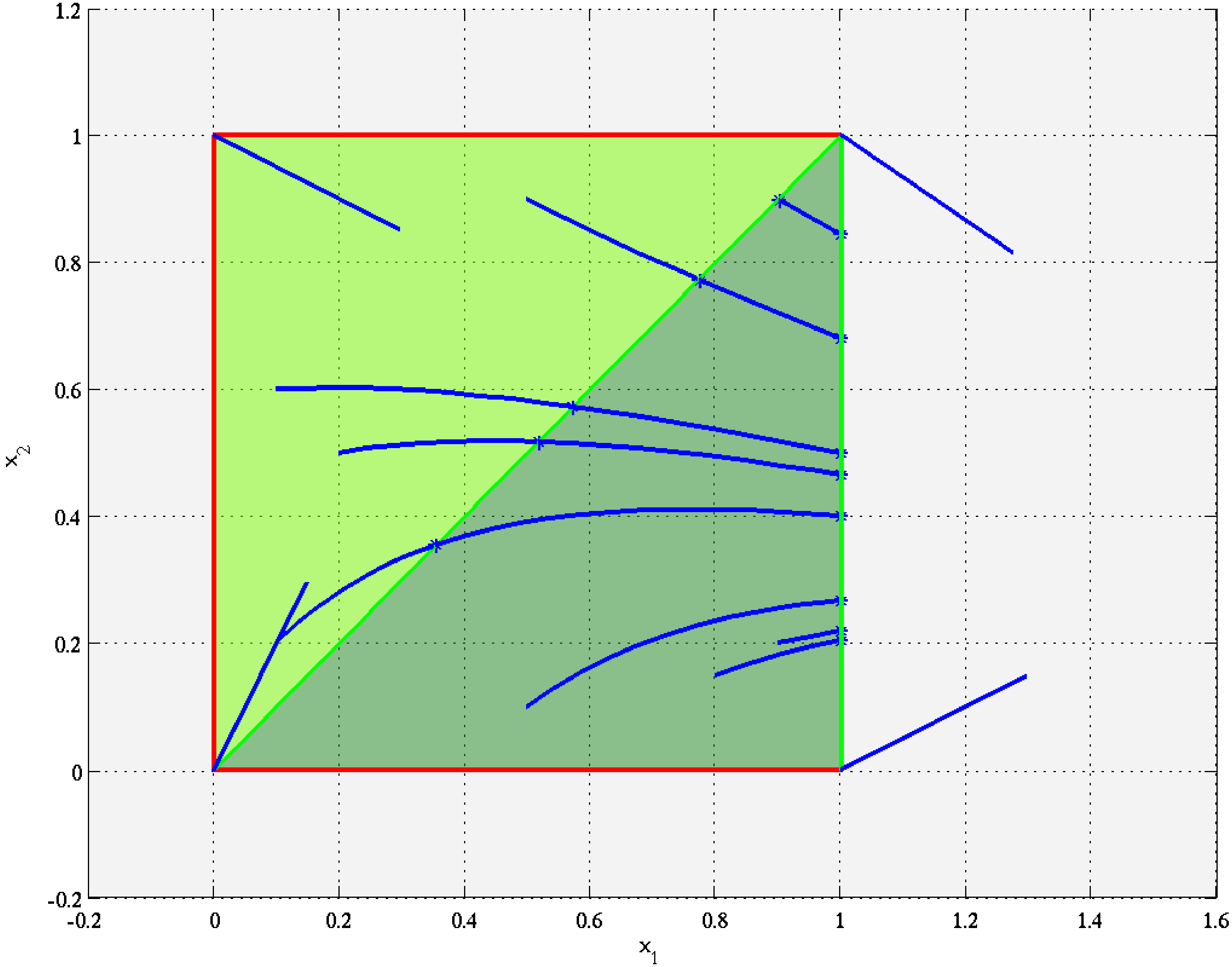
−0.8
−1
−1
−0.5
0
0.5
1
1.5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1
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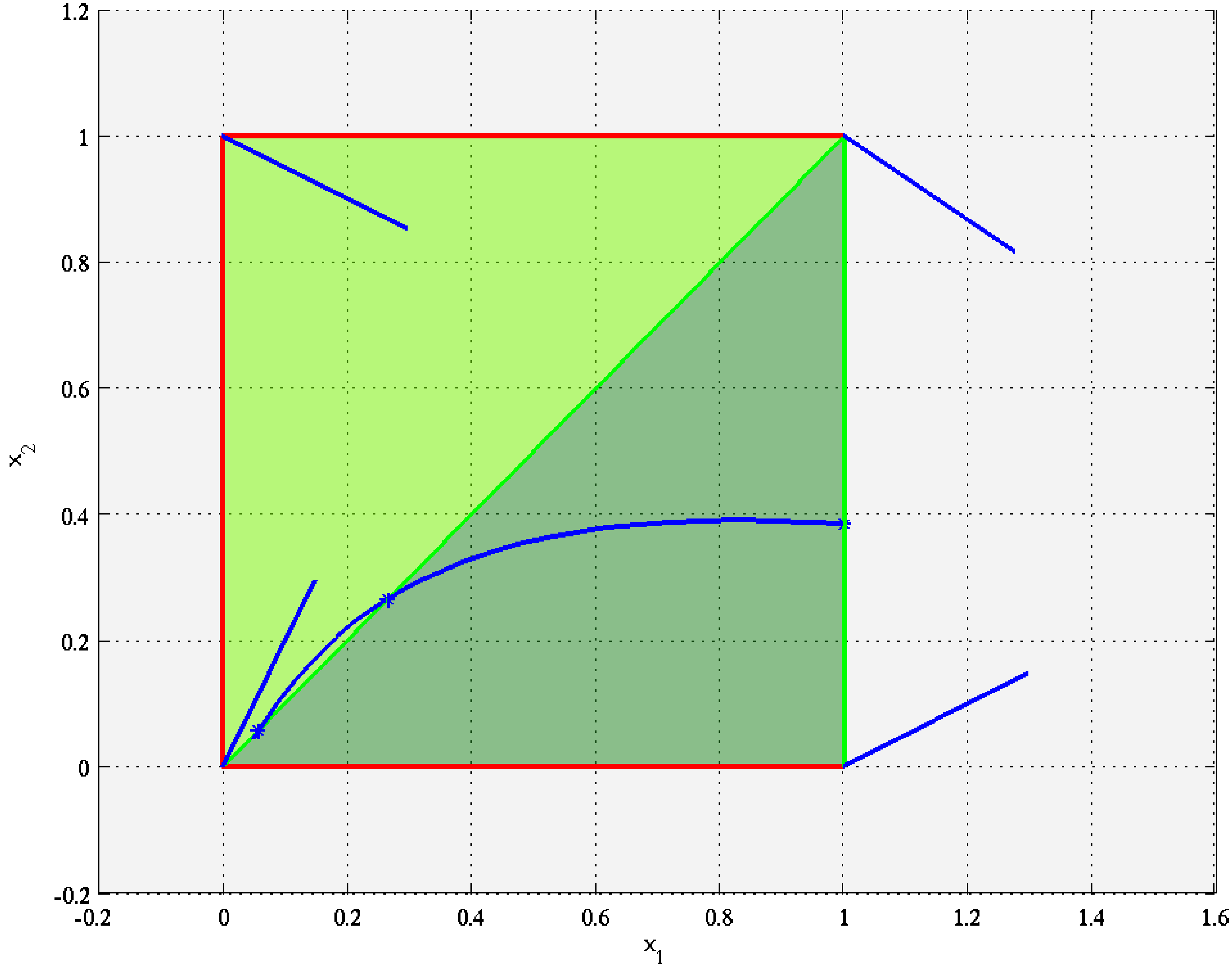
1.4
1.6
x_2
x_1



1.2
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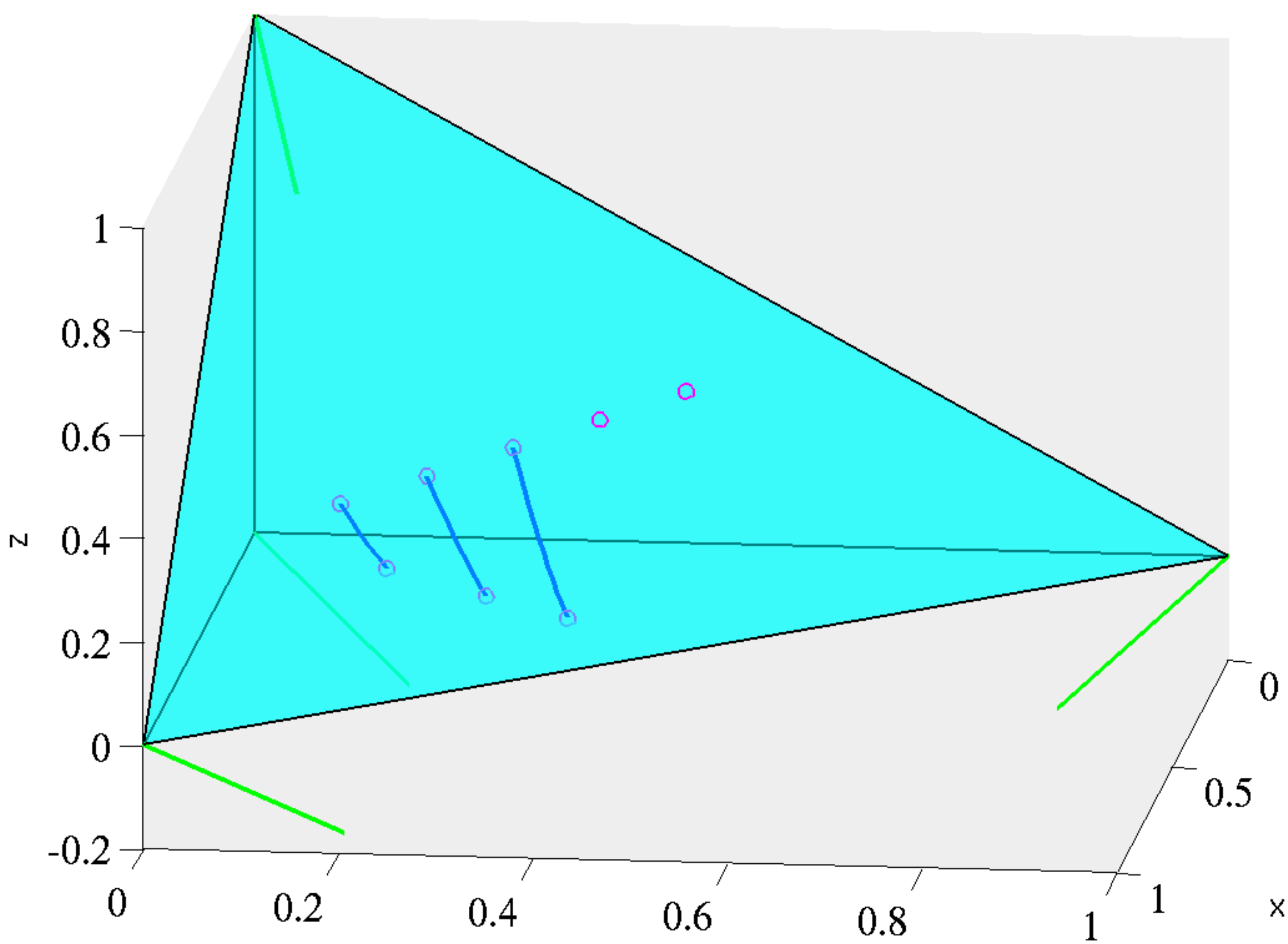
x_1
x_2


z
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0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
0
0.5
1
x

Problem Stabilization to a fixed point

Consider the affine system on a simplex,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + a, \; x(t_0) = x_0, \\ & X \subset \mathbb{R}^N \text{ simplex}, \; U \subset \mathbb{R}^m \text{ polytope}, \; x_f \in X \text{ fixed point.} \end{aligned}$$

Determine an affine control law,

$$k(x) = Fx + g,$$

such that,

1. control law k is admissible: $\forall x \in X$, $k(x) \in U$;
2. state trajectory is admissible: $\forall t \in T$, $x(t, x_0) \in X$;
3. state trajectory converges to fixed point: $\lim_{t \to \infty} x(t, x_0) = x_f$.

Theorem Stabilization-to-fixed-point

Problem is solvable if and only if

$$\exists\ u_1, \ldots, u_{N+1} \in U, \text{ such that}$$

$$(1) \qquad n_i^T(Av_j + Bu_j + a) \le 0,\ \forall j \in \mathbb{Z}_{N+1},\ \forall i \in \mathbb{Z}_{N+1} \backslash \{j\};$$

$$(2) \qquad B \sum_{j=1}^{N+1} \mu_j v_j = -Ax_f - a; \text{ where } x_f = \sum_{j=1}^{N+1} \mu_j v_j;$$

$$(3) \qquad \mathrm{span}(\{Av_j + Bu_j + a | \forall j \in \mathbb{Z}_{N+1}\}) = \mathbb{R}^N.$$

Def. Discrete-event system

$$DES = (Q, E, f),\ Q \text{ state set, } E \text{ event set,}$$

$$f : \mathrm{Dom}(f) \subset Q \times E \to Q \text{ transition function,}$$

$$E(q) = \{e \in E | (q, e) \in \mathrm{Dom}(f)\},\ \text{subset of eligible events, } \forall q \in Q,$$

$$(q_0, q_1, \ldots, q_n),\ q_i = f(q_{i-1}, e_i).$$

For control law k_q define,

$s : Q \to E$ **supervisor**, $\forall q \in Q \;\; s(q) \subseteq E(q)$,

$e \in s(q)$ if there exists a state trajectory

x_q which leaves X_q through guard $G_q(e)$.

Problem **Reach-avoid problem for a PAHS**.

$$
\begin{aligned}
DES &= (Q, E, f), \\
&\quad Q_s \subset Q \text{ starting states, } q_t \in Q \text{ target state, } Q_u \in Q \text{ unsafe states.}
\end{aligned}
$$

Determine a supervisor S such that,

1. S/DES is nonblocking except, possibly, at the terminal state;
2. $q_0 \in Q_s$ and there exists an integer $n \in \mathbb{Z}_+$ such that $q_n = q_t$;
3. $\forall i \in \mathbb{N}_n$, $q_i \notin Q_u$.
4. **Reach-avoid-stay** $s_{q_t} = \emptyset$.
5. **Reach-avoid-converge** $\lim_{t \to \infty} x_{q_t}(t) = x_f$.

Remark The supervisor at $q \in Q$ determines a control law k_q for the affine system on a simplex.

Algorithm Reach-avoid problem (is reachability of DES systems)

1. If $q_t \in Q_u$ then terminate else $Q_0 = \{q_t\}$.
2. While not $(Q_s \subset Q_i$ or $Q_i = Q_{i-1})$ do $i = i + 1$,

$$
\begin{aligned}
(2.1) \qquad & Q_i = Q_{i-1} \cup \\
& \cup \{q \in Q \backslash Q_u | \exists s(q) \subseteq E(q) \ \ \forall e \in s(q), \ \ f(q,e) \subset Q_{i-1}\};
\end{aligned}
$$

(2.2) $\qquad S(q)$ local supervisor found in Step (2.1) (Control law k_q)

Output: Q_i, $(Q_s \subseteq Q_i)$, $(Q_i = Q_{i-1})$, and $\{s(q) \in S(q), q \in Q_i\}$.

Theorem Reach-avoid problem
If Algorithm terminates with $Q_s \subseteq Q_j$
then there exists a solution to Problem Reach-avoid.

Remarks (1) Sufficient condition for reachability.
(2) Computationally tractable.

Example Reach-avoid-stabilize

$$
\begin{aligned}
\dot{x}_q(t) &= A(q)x_q(t) + B(q)u(t) + a(q),\ x_q(t_0) = x_0,\\
Q &= \{q_1, \ldots, q_5\},\ E = \{e_1, \ldots, e_5\},\\
X_q &= \left\{x \in \mathbb{R}^2 | x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\right\},\ \forall q \in Q,\\
G_{q_1}(e_2) &= F_3,\ G_{q_1}(e_3) = F_2,\ G_{q_1}(e_4) = F_1,\ \text{etc.}\\
\dot{x}_{q_1} &= \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} x_{q_1}(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \text{etc.}\\
Q_s &= \{q_1\},\ q_t = q_5,\ Q_u = \{q_4\}.
\end{aligned}
$$

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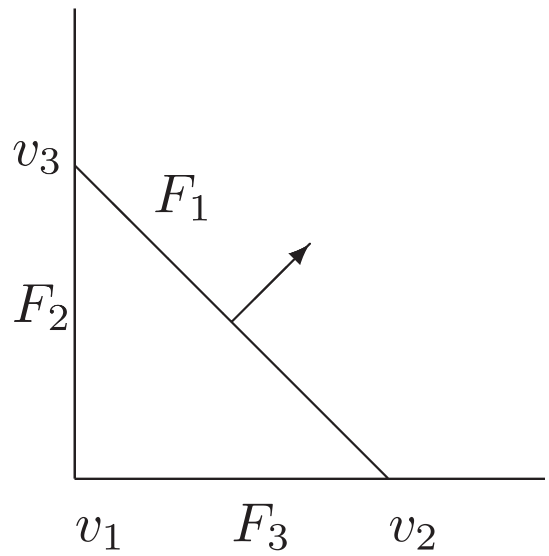

Example (Continued)

Control law specified by:

$$
\begin{aligned}
k_{q_1}(x) &= 1, \quad q_1 \mapsto q_2, \\
k_{q_2}(x) &= 0, \quad q_2 \mapsto \{q_3, q_5\}, \\
k_{q_3}(x) &= 1, \quad q_3 \mapsto q_5, \\
k_{q_5}(x) &= -x_1 - \frac{3}{4}x_2 + \frac{1}{2}, \quad q_5 \mapsto q_5.
\end{aligned}
$$

Remarks on control synthesis for PAHS

- Control synthesis for PAHS on polytopes.
- Computer program package for control of PAHS under construction. (Margreet Nool) (Polyhedral library (K. Fukuda) via ETH.MPT)
- Control-to-facet also developed for a multi-affine system on a rectangle. (C. Belta, L.C.G.J.M. Habets, V. Kumar (2003)).
- Optimal control for control-to-facet.
- Partial observations for control of PAHS.
- Robustness aspects.
- Application to automotive control.
  Control of the idle speed of a car engine
  (A. Balluchi etal. (Parades), JHvS (HSCC.2004)).

Realization theory - Overview of results

- Reduction of affine systems on polytopes due to unobservability. (LH, JHvS (MTNS.2002)).
- Undecidability of observability of piecewise-affine hybrid systems. (PC, JHvS (CDC.2004)).
- Sufficient conditions for observability of piecewise-affine hybrid systems. (PC, JHvS (HSCC.2004)).
- Realization of linear switched systems - A power series approach. (MP (MTNS.2004)).
- Realization of linear hybrid systems. (MP (2005)).
- Realization of bilinear hybrid systems. (MP (2005)).

Computability for hybrid systems

- Approximability in the space of trajectories of hybrid systems (PJC (MTNS.2004)).
- Computability of the reachable and chain reachable sets of dynamic systems (PJC (submissions)).
- Computational model based on type-two Turing machines and computable topology/analysis (K. Weihrauch).

Concluding remarks

CWI Results

1. Control synthesis of PAHS:
   (1) Control-to-an-admissible-exit-facet.
   (2) Control-to-exit.
   (3) Stabilization-to-a-fixed-point.
2. Realization of subclasses of hybrid systems.
3. Computability for dynamic systems.

CWI Research plan

1. Control of PAHS systems, both at continuous and at discrete level.
2. Computer programs for control of PAHS.
3. Realization of hybrid systems.
4. Computability of hybrid systems.

Research in hybrid systems

- More experience with examples of hybrid systems.
- System theory and realization.
- Computability of problems of hybrid systems.
- Control synthesis:
  (a) Control at the discrete-event level of PAHS.
  (b) Control at the continuous level of PAHS.
- Control of networks of hybrid systems.

Acknowledgements

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The end!