Approximation of Reachable Sets

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Overview

- Discretisation of systems.
- Reachability analysis and optimal control.
- Open and closed systems.
- Chain reachability and computability.
- Conley index techniques

References

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Discrete time systems

- Consider system properties of

$$x_{n+1} = f(x_n, u_n, v_n)$$

  where $x_n \in X$ is the state, $u_n \in U$ is the control input, and $v_n \in V$ is noise.
- For noise-free systems, represent as a *mulitvalued map* $F : X \twoheadrightarrow X$ given by $F(x) = f(x, U)$.
- A control law can be represented by a system $G : X \twoheadrightarrow X$ with $G(x) \subset F(x)$.
- The *graph* of $F : X \twoheadrightarrow X$ is the set

$$\mathrm{Graph}(F) = \{(x, y) \in X \times X : y \in F(x)\}.$$

State-space discretisations

- Let $\mathcal{P}$ be a finite cover of X by compact sets (e.g. a partition into boxes).
- The system is represented by a directed graph G with vertices $\mathcal{P}$.
- Reachability properties can be determined by finding paths using Dijkstra's shortest path algorithm.
- Typically, try to obtain properties in the limit

$$\mathrm{diam}(\mathcal{P}) := \sup\{\mathrm{diam}(P) : P \in \mathcal{P}\} \to 0.$$

Upper and lower discretisations

- Write $P \to Q$ if
$$(\exists x \in P)\,(\exists y \in Q)\;(\exists u \in U)\;(f(x,u) = y).$$
- If there is a trajectory (x_i) with $x_i \in P_i$, then $P_0 \to P_1 \to \cdots \to P_n$.
- Write $P \Rightarrow Q$ if
$$(\forall x \in P)(\exists y \in Q)\;(\exists u \in U)\;(f(x,u) = y).$$
- If $P_0 \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_n$, then there is a trajectory (x_i) with $x_i \in P_i$.
- Could also consider
$$(\forall y \in Q)\,(\exists/\forall x \in P)\;(\exists u \in U)\;(f(x,u) = y)$$

Optimal control

- Take a continuous *cost function* $q : X \times U \to \mathbb{R}^+$.
- Cost of trajectory $\mathbf{x} = (x_n)$ starting at x given by control $\mathbf{u} = (u_n)$ is

$$J(x, \mathbf{u}) = \sum_{n=0}^{N-1} q(x_n, u_n)$$

- Consider the optimal control problem:

Minimise $J(x, \mathbf{u})$ such that $x_0 = x$ and $x_N \in S$

for some target set S.

- The *value function* $V : X \to R^+$ is given by

$$V(x) = \inf\{J(x, \mathbf{u}) : x_0 = x \text{ and } x_N \in S\}$$

Discretisation of cost estimates

- For $x, y \in X$, define

$$w(x,y) = \inf_{u \in U} \{q(x,u) : f(x,u) = y\}.$$

- For $P, Q \subset X$, define

$$\underline{w}(P,Q) = \inf_{x \in P} \inf_{y \in Q} w(x,y), \text{ and } \overline{w}(P,Q) = \sup_{x \in P} \inf_{y \in Q} w(x,y).$$

- It is clear that

$$\underline{w}(P,Q) < \infty \text{ iff } P \rightarrow Q, \text{ and } \overline{w}(P,Q) < \infty \text{ iff } P \Rightarrow Q.$$

- More generally, a *weight function* is a function

$$w : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}^+ \cup \{\infty\}.$$

Trajectory cost estimates

- If $\vec{P} = (P_i)$ is a discrete trajectory, then let

$$J^w(\vec{P}) = \sum_{n=0}^{N-1} w(P_n, P_{n+1})$$

- Let $\underline{J}(\vec{P}) = J^{\underline{w}}(\vec{P})$ and $\overline{J}(\vec{P}) = J^{\overline{w}}(\vec{P})$
- Define $V^w_{\mathcal{P}} : X \to R^+$ by

$$V^w_{\mathcal{P}}(x) = \inf\{J^w(\vec{P}) : x \in P_0 \text{ and } P_N \cap S \neq \emptyset\}$$

- Let $\underline{V}_{\mathcal{P}}(x) = V^{\underline{w}}_{\mathcal{P}}$ and $\overline{V}_{\mathcal{P}} = V^{\overline{w}}_{\mathcal{P}}$.
- Clearly $V^{\underline{w}}_{\mathcal{P}}(x) < V(x) < V^{\overline{w}}_{\mathcal{P}}$.
- If S is reachable from x, then $V^{\underline{w}}_{\mathcal{P}}(x) < \infty$, and if $V^{\overline{w}}_{\mathcal{P}} < \infty$, then S is reachable from x.

Compact systems

- A system is *compact* if the graph of $F : X \twoheadrightarrow X$ is a compact set.
- A map $f : X \times U \to X$ is a compact system if X and U are compact sets.
- **Theorem** *If f is a compact system $V(x) < \infty$ (i.e. S is reachable from x), then*

$$\underline{V}_{\mathcal{P}}(x) \to V(x) \text{ as } \operatorname{diam}(\mathcal{P}) \to 0.$$

[Oliver Junge and Hinke Osinga, “A Set Oriented Approach to Global Optimal Control”, preprint.]

Open systems

- A system is *open* if the graph of $F : X \twoheadrightarrow X$ is an open set.
- A map $f : X \times U \to X$ gives rise to an open system if U is open and $\frac{\partial f}{\partial u}$ has full row rank.
- Typical systems are not open, but the n-step system may be for some $n > 1$.
- If F is an open system and $y \in F(x)$, then if $x \in P$, $y \in Q$, and the diameter of Q is sufficiently small, then $F(P) \supset Q$.
- **Theorem** *If f is an open system and $V(x) < \infty$, then*

$$\overline{V}_{\mathcal{P}}(x) \to V(x) \text{ as } \operatorname{diam}(\mathcal{P}) \to 0.$$

Approximate systems

- For $\epsilon > 0$ define $\widehat{F}_\epsilon$ to be

$$\widehat{F}_\epsilon(x) = \{y \in X : \exists\, x', y' \in X \text{ such that } y' \in F(x') \text{ and } d(x, x') < \epsilon/2 \text{ and } d(y, y') < \epsilon/2\}$$

- The graph of $\widehat{F}_\epsilon$ is an open set, so $\widehat{F}_\epsilon$ is an open system.
- An orbit of $\widehat{F}_\epsilon$ is an ϵ-*chain* for F.
- S is *chain reachable* from x if there is an ϵ-chain from x to y for all $\epsilon > 0$.

Discretisations of approximate systems

- Fix $\epsilon > 0$, and consider a cover by sets of diameter less than ϵ.
- Suppose $x \in P$ and $d(x,y) < \epsilon/2$ for all $y \in P$.
- Then if $F(x) \cap Q \neq \emptyset$, we must have $\widehat{F}_\epsilon(y) \cap Q \neq \emptyset$ for all $y \in P$.
- Hence a lower discretisation of $\widehat{F}_\epsilon$ can be rigorously computed.
- Therefore, there is a graph such that

$$(P \to Q) \;\Rightarrow\; (\forall x \in P)\,(\exists y \in Q)\;(\exists u \in U)\;(f(x,u) = y).$$

Chain reachability vs reachability

- We have an algorithm to prove $\mathrm{Reach}(x) \cap S = \emptyset$ and $\mathrm{ChainReach}(x) \cap S \neq \emptyset$.
- Unfortunately, for general systems, $\mathrm{Reach}(x) \neq \mathrm{ChainReach}(x)$.
- Thus the theory for chain-reachable sets is very different from the theory of reachable sets.
- **Conjecture** *Let A be a stable chain-transitive set for the noise-free system F. Then there exists $\delta > 0$, and a control law $x \in G(u)$ such that for all $\epsilon > 0$, if x and y are points of A, then the orbit (x_n) with $x_0 = x$ reaches the ϵ-neighbourhood of y with probability 1.*

  [Michel Benaïm & Morris W. Hirsch, “Asymptotic pseudotrajectories and chain recurrent flows, with applications”, *J. Dynam. Differential Equations* **8** (1996), 141–176.]

Chain transitivity

- A subset A of X is *chain transitive* for a compact system F if for all $x, y \in A$, there is an ϵ-chain from x to y for any $\epsilon > 0$.
- A maximal chain transitive set is a *chain component*.
- If G is a contol law then the chain components of G are subsets of those of F.
- Chain components and chain reachability relations between them can be computed using the *Conley index*.

Computability

- For even simple classes of systems (e.g. piecewise-constant derivative) systems, reachability / controllability is uncomputable.

  [Eugene Asarin, Oded Maler & Amir Pnueli, “Reachability analysis of dynamical systems having piecewise-constant derivatives”, *Theor. Comp. Sci.* **138** (1995) 35–65.]

- For open systems, the reachability/controllability properties should be recursively computable by taking finer partitions.
- For compact systems, should be able to recursively compute non-controllable and chain-controllable sets.

Conley index and invariant sets

- A set is if *isolated invariant* if it is the maximal invariant set in a neighbourhood of itself.
- Many system properties, including chain recurrent sets and attractors, can be expressed in terms of isolated invariant sets.
- Isolated invariant sets and their structure, and hence global system properties, can be computed using the Conley index.
- The Conley index may be able to (partially) bridge the gap between reachability and chain-reachability. Charles Conley. “Isolated Invariant Sets and the Morse Index,” AMS, 1978.

Conclusions

- State space discretisations give a computational approach to determining (optimal) controls.
- System-theoretic properties can be analysed in terms of discretisations, and computability properties studied.
- Conley index theory provide a further tool for analysis of system properties.