Robust Constrained Optimal Control

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Summary of Previous Results…

Linear Quadratic Regulator

$$\min_U \quad x(N)'Px(N) + \sum_{k=0}^{N-1} x(k)'Qx(k) + u(k)'Ru(k)$$
$$\text{subj.to} \quad x(k+1) = Ax(k) + Bu(k),$$

$$P, Q \succeq 0, \quad R \succ 0, \quad x(k) \in \mathrm{R}^n, \quad u(k) \in \mathrm{R}^m$$

$$U \triangleq \{u(0), u(1), \ldots, u(N-1)\}$$

Solution: $u^*(k) = K(k)x(k), \quad k = 0, \ldots, N-1$

where *K(k)* is found from Difference Riccati Equation

Extension 1: Constraints

$$\min_U \quad x(N)'Px(N) + \sum_{k=0}^{N-1} x(k)'Qx(k) + u(k)'Ru(k)$$

$$\text{subj.to} \quad x(k+1) = Ax(k) + Bu(k),$$

$$Ex(k) + Lu(k) \leq M, \quad k = 0, \ldots, N-1$$

$$x(N) \in \mathcal{X}_f$$

$x(k) \in \mathrm{R}^n, \quad u(k) \in \mathrm{R}^m$

$P, Q \succeq 0, \quad R \succ 0$

$U \triangleq \{u(0), u(1), \ldots, u(N-1)\}$

$\mathcal{X}_f$ polyhedral set

Feedback solution?

How do we compute it?

Extension 2: Cost based on 1/∞-norm

$$\min_U \quad ||Px(N)||_p + \sum_{k=0}^{N-1} ||Qx(k)||_p + ||Ru(k)||_p$$

$$\text{subj.to} \quad \begin{array}{l} x(k+1) = Ax(k) + Bu(k), \\ Ex(k) + Lu(k) \leq M, \quad k = 0, \ldots, N-1 \\ x(N) \in \mathcal{X}_f \end{array}$$

$$x(k) \in \mathrm{R}^n, \quad u(k) \in \mathrm{R}^m, \qquad U \triangleq \{u(0), u(1), \ldots, u(N-1)\}$$

$$v \in \mathrm{R}^n, H \in \mathrm{R}^{n \times n} \rightarrow ||Hv||_2 \triangleq vHv$$

$$P, Q \succeq 0, \quad R \succ 0$$

$$v \in \mathrm{R}^n, H \in \mathrm{R}^{m \times n} \rightarrow ||Hv||_\infty \triangleq max\{|H_1 v|, \ldots, |H_m v|\}$$

$$v \in \mathrm{R}^n, H \in \mathrm{R}^{m \times n} \rightarrow ||Hv||_1 \triangleq |H_1 v| + \ldots + |H_m v|$$

$$P, Q, R \ \text{ full coloumn rank}$$

Feedback solution?

How do we compute it?

Extension 3: PWA Dynamical System

$$\min_U \quad ||Px(N)||_p + \sum_{k=0}^{N-1} ||Qx(k)||_p + ||Ru(k)||_p$$

$$\text{subj.to} \quad x(t+1) = A_i x(t) + B_i u(t) + f_i \quad \text{if } [x(t), u(t)] \in \mathcal{X}_i, \quad i = 1, \ldots, s$$

$$Ex(k) + Lu(k) \leq M, \quad k = 0, \ldots, N-1$$

$$x(N) \in \mathcal{X}_f$$

$$x(k) \in \mathrm{R}^n, \quad u(k) \in \mathrm{R}^m, \quad U \triangleq \{u(0), u(1), \ldots, u(N-1)\}$$

Feedback solution?

How do we compute it?

Tool: Multiparametric Programming

The mathematical program

$$\min_{\varepsilon} \quad \varepsilon^T H \varepsilon + f^T \varepsilon$$
$$\text{s.t.} \;\; G\varepsilon \leq w + Fx$$

is solved for all *x* by using

$\varepsilon \in \mathrm{R}^{n_c} \times \{0, 1\}^{n_b}$

- H≠ 0 ⇒ multiparametric Mixed Integer QP (mp-MIQP)
- H= 0 ⇒ multiparametric Mixed Integer LP (mp-MILP)

$\varepsilon \in \mathrm{R}^{n}$

- H≠ 0 ⇒ multiparametric QP (mp-QP)
- H= 0 ⇒ multiparametric LP (mp-LP)

to compute

$$\varepsilon^*(x) = f_{PWA}(x)$$

Multiparametric Program Solvers

• mp-QP	Bemporad, Morari, Dua, Pistikopoulos, 2000 Godwin, De Dona', 2000 Tondel, Johansen, Bemporad, 2001
• mp-LP	Gal, 1972 Borrelli, Bemporad, Morari, 2000
• mp-MILP	Dua, Pistikopoulos, 1999
• mp-MIQP	Borrelli, Baotic, Bemporad, Morari, 2002 Dua, Pistikopoulos, 2002

Optimal Control Problems: Results

- Unconstrained Linear System

Solution: $u^*(k) = K(k)x(k)$

where gain *K(k)* is found from Riccati Equation

- Constrained Linear System

Solution: $u^*(k) = F_i^k x(k) + G_i^k \text{ if } x(k) \in \mathcal{D}_i^k$

where polyhedra $\mathcal{D}_i^k$, controller F_i^k, G_i^k are found from
mp-LP (linear performance index)
mp-QP (quadratic performance index)

- Constrained PWA System

Solution: $u^*(k) = F_i^k x(k) + G_i^k \text{ if } x(k) \in \mathcal{D}_i^k$

where sets $\mathcal{D}_i^k$, controller F_i^k, G_i^k are found from
mp-MILP (linear performance index)
mp-MIQP (quadratic performance index)

…New Extension…

Problem Setup

- Linear Uncertain Model

$$x(t+1) = A(w(t))x(t) + B(w(t))u(t) + Ev(t)$$
$$A(w) = A_0 + \sum_{i=1}^{q} A_i w_i, \quad B(w) = B_0 + \sum_{i=1}^{q} B_i w_i$$

- Constraints:

$$Fx(t) + Gu(t) \leq f$$
$$v(t) \in \mathcal{V} \quad \mathcal{V} \text{ bounded polyhedron}$$
$$w(t) \in \mathcal{W} \quad \mathcal{W} \text{ bounded polyhedron}$$

- Goal:

Find a feedback controller guaranteeing
constraint fulfillment for all disturbances
and optimal worst-case performance

see also: Scokaert, Mayne, '98, J.H. Lee and Z.Yu, '97

Open-Loop
Constrained Robust Optimal Control

MIN over the control input of the MAX over the possible disturbances

$$J_N^*(x_0) \triangleq \min_{u_0,\ldots,u_{N-1}} J(x_0, U)$$

$$\text{subj. to} \quad \left\{ \begin{array}{l} Fx_k + Gu_k \leq f \\ x_{k+1} = A(w_k)x_k + B(w_k)u_k + Ev_k \\ x_N \in \mathcal{X}^f \\ k = 0, \ldots, N-1 \end{array} \right\} \quad \begin{array}{l} \forall v_k \in \mathcal{V}, w_k \in \mathcal{W} \\ \forall k = 0, \ldots, N-1 \end{array}$$

$$J(x_0, U) \triangleq \max_{\substack{v_0,\ldots,v_{N-1} \\ w_0,\ldots,w_{N-1}}} \left\{ \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p + \|Px_N\|_p \right\}$$

$$\text{subj. to} \quad \left\{ \begin{array}{l} x_{k+1} = A(w_k)x_k + B(w_k)u_k + Ev_k \\ v_k \in \mathcal{V} \\ w_k \in \mathcal{W}, \\ k = 0, \ldots, N-1 \end{array} \right.$$

Open-Loop vs. Closed-Loop CROC

Open-Loop Prediction Strategy

- The whole disturbance sequence plays first
- The benefits of feedback not exploited in prediction

  $\Rightarrow$ poor performance
- Uncertainty grows over the prediction horizon

  $\Rightarrow$ possible infeasibility

Closed-Loop Prediction Strategy

- Disturbance and input play one move at a time
- $u^*(k)$ is a function of the current state $x(k)$

  $\Rightarrow$ depends on previous controls and **disturbances**

Closed-Loop Constrained Robust Optimal Control

- Solve for j=N-1,…,0

$$J_j^*(x_j) \triangleq \min_{u_j} J_j(x_j, u_j)$$
$$\text{subj. to} \left\{ \begin{array}{l} Fx_j + Gu_j \leq f \\ A(w_j)x_j + B(w_j)u_j + Ev_j \in \mathcal{X}^{j+1} \end{array} \right\} \forall v_j \in \mathcal{V}, w_j \in \mathcal{W}$$

$$J_j(x_j, u_j) \triangleq \max_{v_j \in \mathcal{V},\ w_j \in \mathcal{W}} \{\|Qx_j\|_p + \|Ru_j\|_p + J_{j+1}^*(A(w_j)x_j + B(w_j)u_j + Ev_j)\}$$

- Where

$$\mathcal{X}^j = \{x \in \mathbb{R}^n | \ \exists u, \ \left(Fx + Gu \leq f, \text{ and } A(w)x + B(w)u + Ev \in \mathcal{X}^{j+1} \ \forall v \in \mathcal{V}, w \in \mathcal{W}\right)\}.$$

- With boundary condition

$$J_N^*(x_N) = \|Px_N\|_p$$
$$\mathcal{X}^N = \mathcal{X}^f,$$

Main Results

Structure

Theorem 1: The solution to the Closed-Loop Constrained Robust Optimal Control problem is a PWA state feedback control law of the form $u^*(x(k)) = F_i^k x(k) + G_i^k$

$$\text{if} \quad x(k) \in \mathcal{P}_i^k \triangleq \{x : M_i^k(j)x \leq K_i^k(j)\}$$

$\{\mathcal{P}_i^k\}_{i=1}^{N_k}$ is a partition of the set $\mathcal{X}_k^*$ of feasible states *x(k)*.

Solution via mp-LP

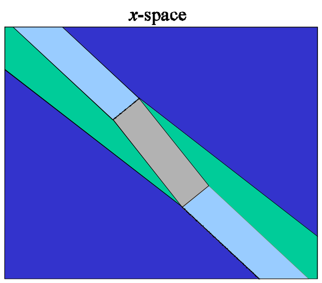
Theorem 2: Closed-Loop Constrained Robust Optimal Control can be solved by solving N mp-LPs

Conclusions

- Closed-Loop Constrained Robust Optimal Control
  - Formulated via Dynamic Programming
  - Exactly solved off-line via Multiparametric Programming (no gridding!)

- The solution to CL-CROC is a piecewise affine state-feedback law

$$u(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad H_1 x \leq K_1 \\ \vdots & \quad \vdots \\ F_N x + G_N & \text{if} \quad H_N x \leq K_N \end{cases}$$



$\Rightarrow$ No need to store the polyhedra (Borrelli, Baotic, Bemporad, Morari, CDC 2001)

- Future extension: Hybrid systems

THE END

CL-CROC - Solution

- The last min-max problem

$$J^*_{N-1}(x_{N-1}) = min_{\ u_{N-1}} \left(||Ru_{N-1}||_p + max_{v_{N-1},w_{N-1}} \ ||Px_N||_p \right)$$

can be solved multipametrically to obtain

> u*(N-1) as piecewise affine and continuous function of x(N-1)
>
> *J*(N-1)* is a piecewise-linear and convex function of *x(N-1)*

- The min-max problem at the previous step

$$J^*_{N-1}(x_{N-1}) = min_{\ u_{N-2}} \left(||Ru_{N-2}||_p + max_{v_{N-2},w_{N-2}} \ J^*(x_{N-1}) \right)$$

can be again solved multiparametrically.

- By iterating the above min-max procedure we obtain

> u*(k) as piecewise affine and continuous function of x(k)

CL-CROC: Different Strategies

(Scokaert, Mayne, '98)

- *CL prediction* by introducing one free input sequence for each extreme point of the set of disturbance sequences

  $\Rightarrow$ problem size *grows exponentially* with the time horizon.

- Solve one Linear Program (infinity-norms) at each time step *t*

  $\Rightarrow$ *exact solution*

  $\Rightarrow$ *on-line* computation

(J.H. Lee and Z.Yu, '97)

- CL prediction via dynamic programming
- *Discretization* of the state space

  $\Rightarrow$ *approximate solution*

  $\Rightarrow$ *"curse of dimensionality"* *("the technique is impratical for all but small size problems")*

  $\Rightarrow$ *off-line* construction of a look-up table

(Bemporad, Borrelli, Morari, ECC 2001)

- CL prediction via dynamic programming
- Off-line computation of the exact (piecewise affine) state-feedback solution via multi-parametric programming