Control from Computer Science

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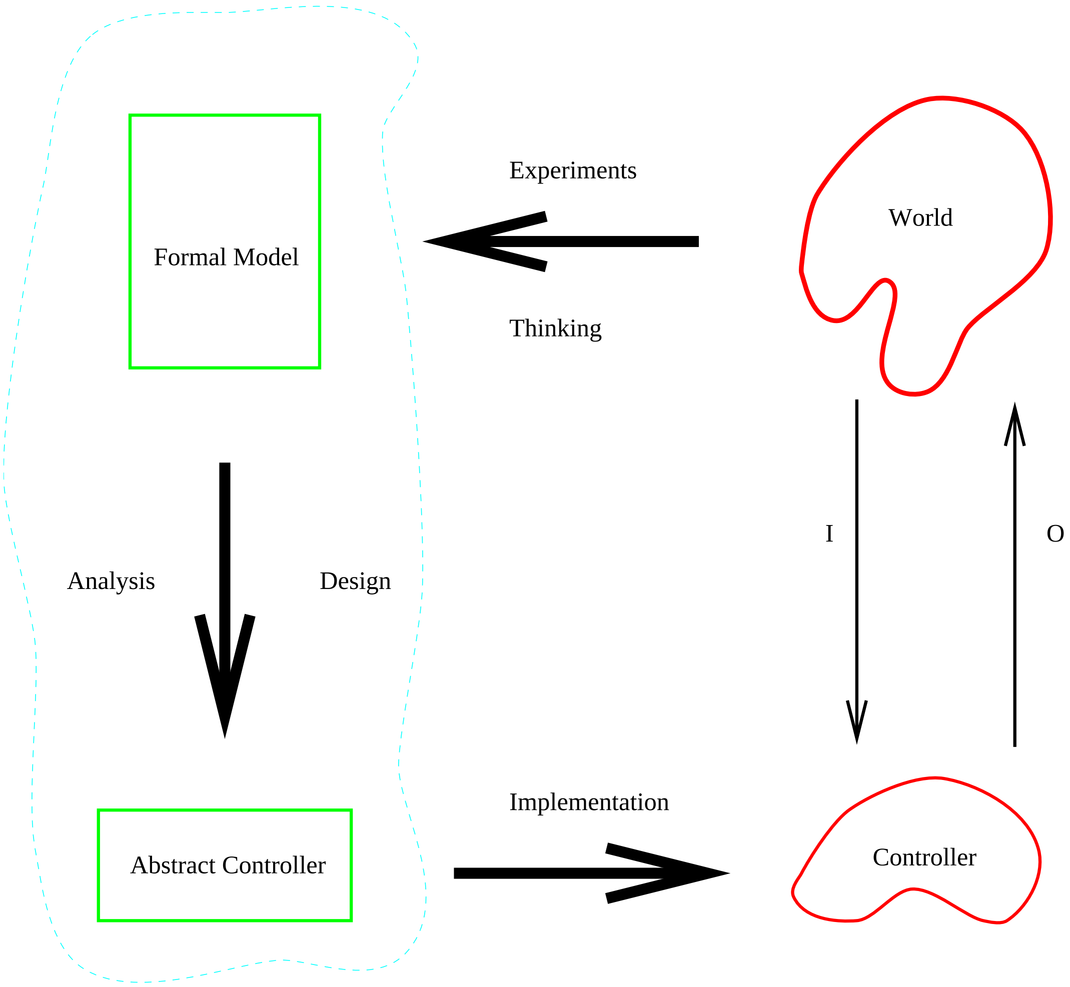
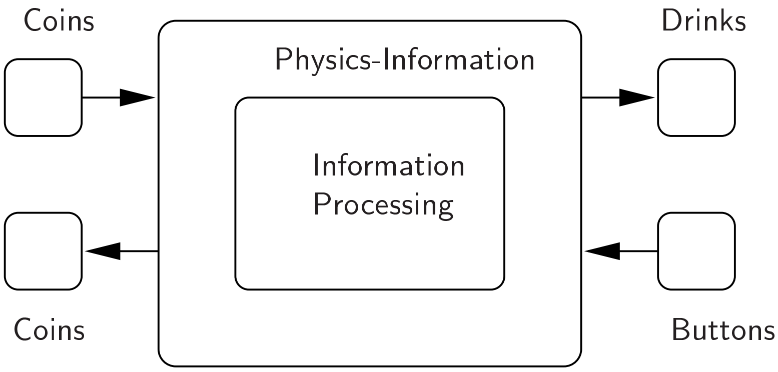
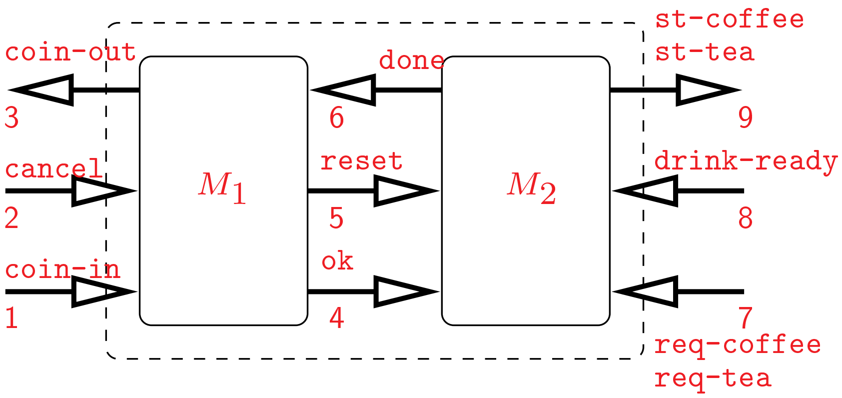
Model-based System Design



The Coffee Machine

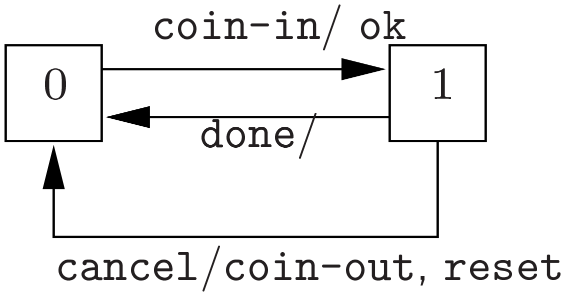




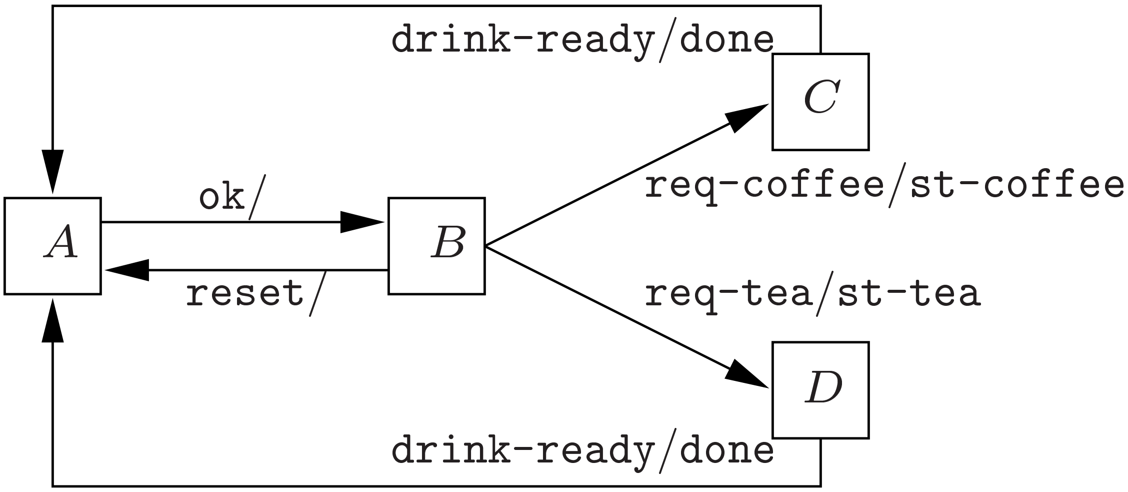
Port	From→To	Event types	Meaning
1	$E \to M_1$	coin-in	a coin was inserted
2	$E \to M_1$	cancel	cancel button pressed
3	$M_1 \to E$	coin-out	release the coin
4	$M_1 \to M_2$	ok	sufficient money inserted
5	$M_1 \to M_2$	reset	money returned to user
6	$M_2 \to M_1$	done	drink distribution ended
7	$E \to M_2$	req-coffee req-tea	coffee button pressed tea button pressed
8	$E \to M_2$	drink-ready	drink preparation ended
9	$M_2 \to E$	st-coffee st-tea	start preparing coffee start preparing tea

The Two Sub-Machines

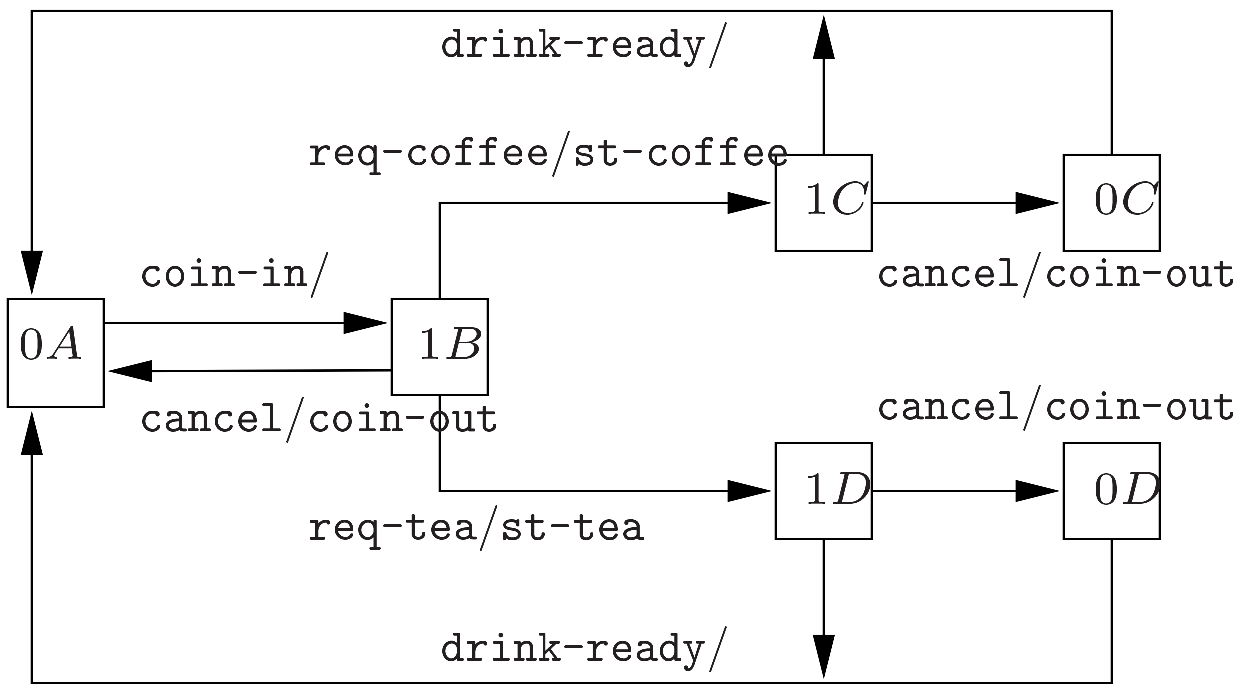
M_1



M_2



The Global Model

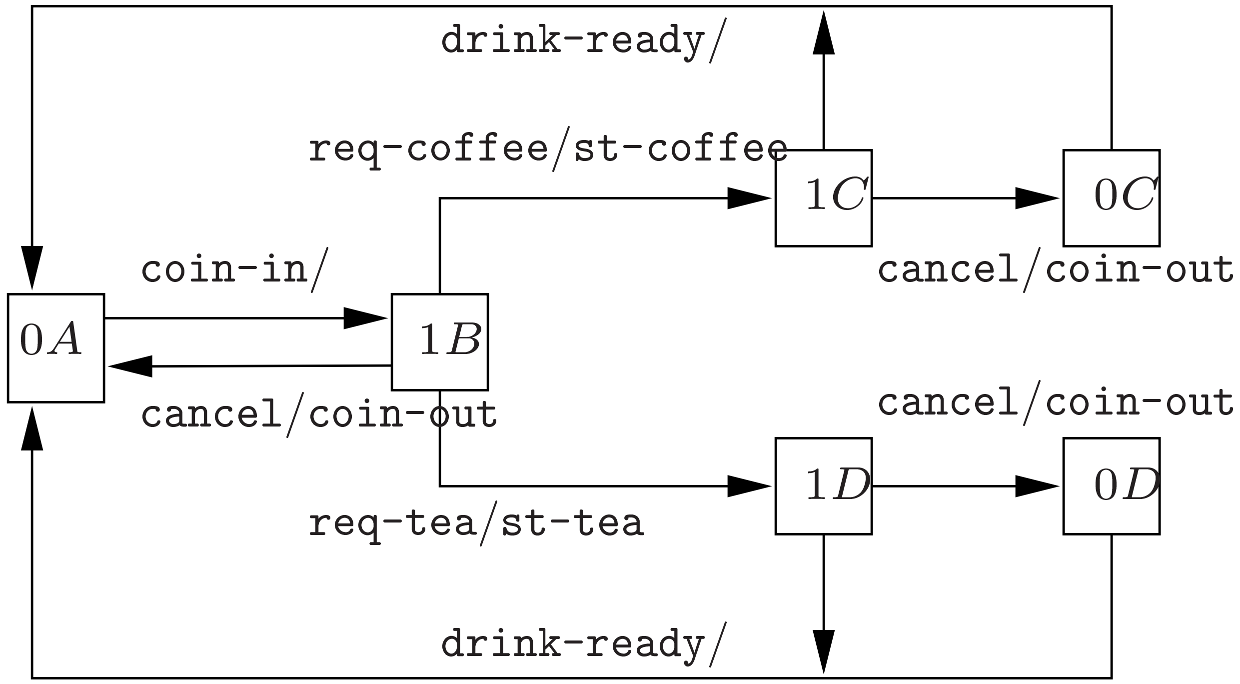


Normal behaviors:

$0A$ coin-in $1B$ cancel coin-out $0A$

$0A$ coin-in $1B$ req-coffee st-coffee
$1C$ drink-ready $0A$

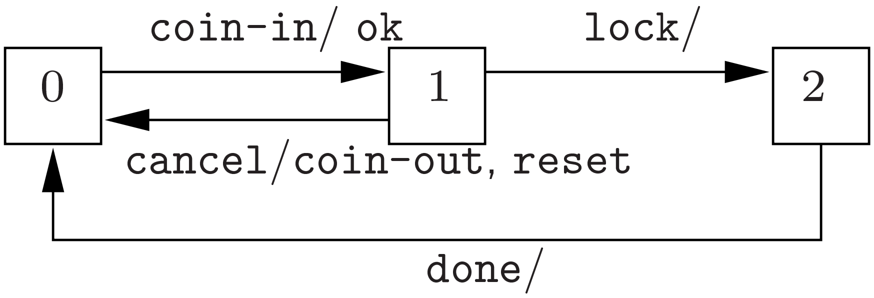
An Unexpected Behavior



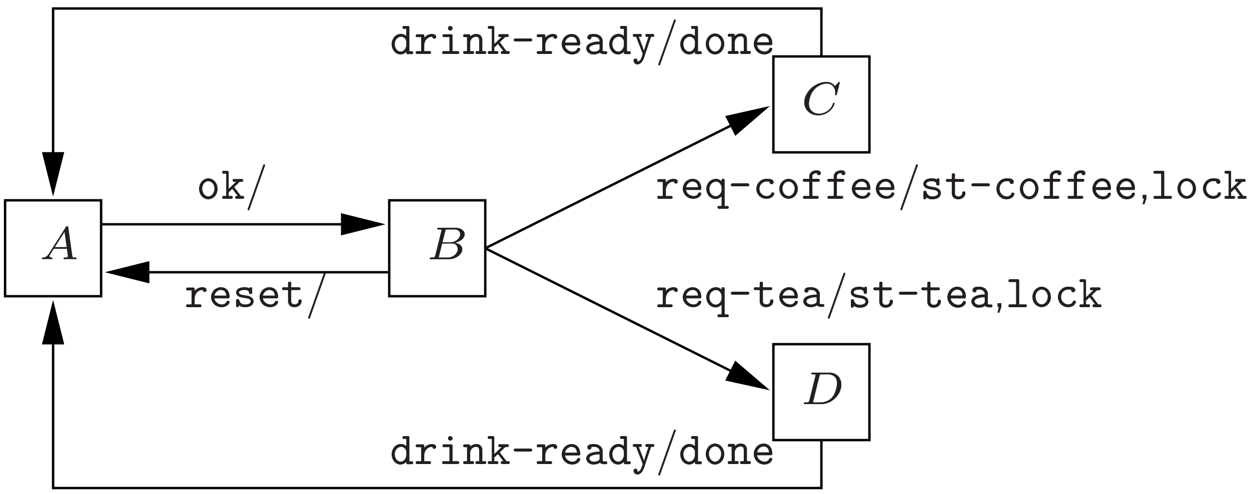
$0A$ coin-in $1B$ req-coffee st-coffee $1C$ cancel coin-out $0C$ drink-ready $0A$

Fixing the Bug

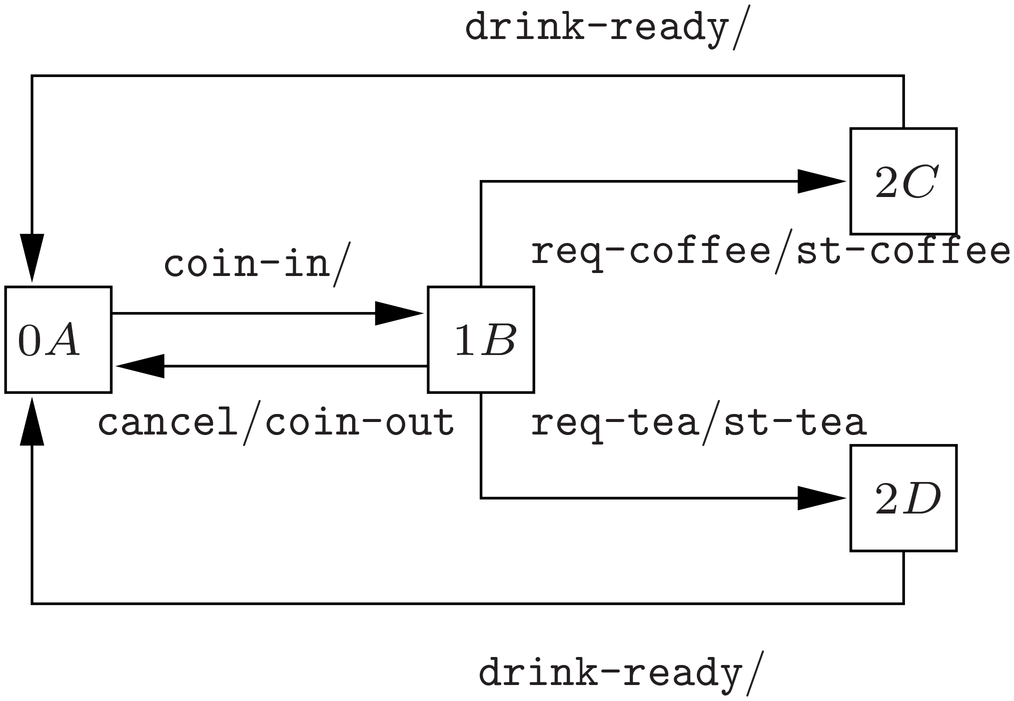
M_1



M_2



Fixing the Bug – the Global Model



The Moral of the Story

1) Many systems can be modeled as a **composition of interacting automata** (transition systems, discrete event systems).

2) Potential behaviors of the system correspond to **paths** in the **global transition graph** of the system.

3) These paths are **labeled** by **input events**. Each input sequence might generate a **different behavior**.

4) We want to make sure that a system responds correctly to **all conceivable inputs.**

5) For every **individual input sequence** we can **simulate** the reaction of the system. But we cannot do it exhaustively due to the huge number of input sequences.

6) Verification is a collection of automatic and semi-automatic methods to analyze **all** the paths in the graph.

7) This is hard for humans to do and even for computers.

Model I: Closed Systems

A transition system is $S = (X, \delta)$ where X is finite and $\delta : X \to X$ is the transition function.

The state-space X has no numerical meaning and no interesting structure.

X^k is the set of all sequences of length k; X^* the set of all sequences.

Behavior: The behavior of S starting from an initial state $x_0 \in X$, is

$$\xi = \xi[0], \xi[1], \ldots \in X^*$$

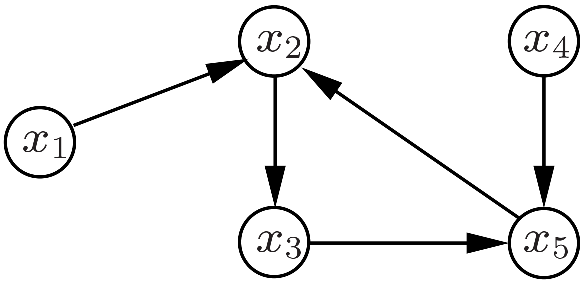
s.t. $\xi[0] = x_0$ and for every i,

$$\xi[i+1] = \delta(\xi[i])$$

Basic Reachability Problem: Given x_0 and a set $P \subseteq X$, does the behavior of S starting at x_0 reach P?

Solution by Forward Simulation

$\xi[0]{:=}x_0$
$F^0 := \{x_0\}$
repeat
  $\xi[k+1]{:=}\delta(\xi[k])$
  $F^{k+1} := F^k \cup \{\xi[i+1]\}$
until $F^{k+1} = F^k$
$F_*{:=}F^k$



$$\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_5\}$$

How to do it for continuous system defined by $\dot{x} = f(x)$?

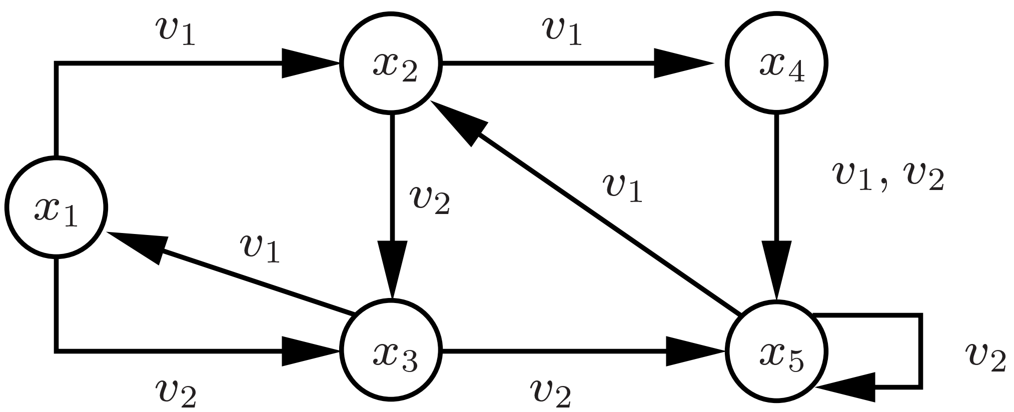
Model II: Systems with One Input

A one-input transition system is $S = (X, V, \delta)$ where X and V are finite $\delta : X \times V \to X$ is the transition function.

Behavior Induced by Input: Given an input sequence $\psi \in V^*$, the behavior of S starting from $x_0 \in X$ in the presence of ψ is a sequence

$$\xi(\psi) = \xi[0], \xi[1], \ldots \in X^* \text{ such that}$$

$$\xi[i+1] = \delta(\xi[i], \psi[i]).$$



$$x_1 \xrightarrow{v_1} x_2 \xrightarrow{v_2} x_3 \xrightarrow{v_2} x_5 \xrightarrow{v_1} x_2 \xrightarrow{v_1} x_4$$

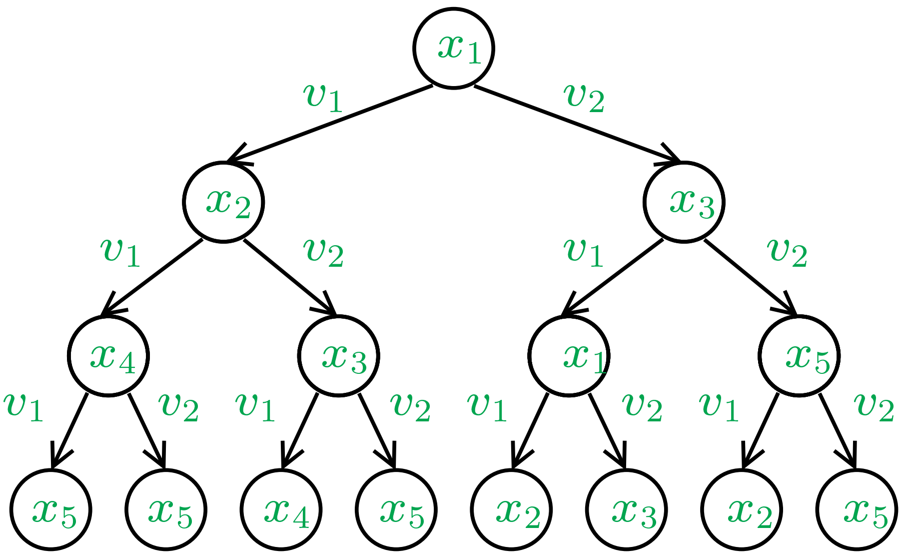
Reachability for Open Systems

The reachability problem: Is there some input sequence $\psi \in V^*$ such that $\xi(\psi)$ reaches P?

For every given ψ we can use the previous algorithm, simulate and obtain $F_*(\psi)$.

For an automaton with n states all states are reachable by sequences of length $< n$.

$$F_* = \bigcup_{\xi \in V^n} F_*(\psi)$$



A More Efficient Way

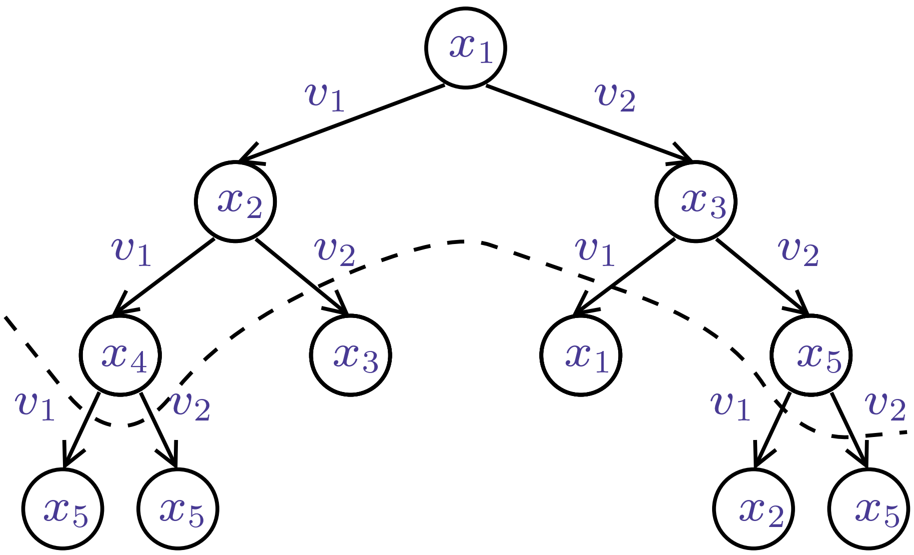
Many different inputs lead to the same state.

Immediate successors: $\delta(x) = \{x' : \exists u \ \delta(x,u) = x'\}$

Successors of a set F: $\delta(F) = \{\delta(x) : x \in F\}$

Forward reachability algorithm (breadth-first):

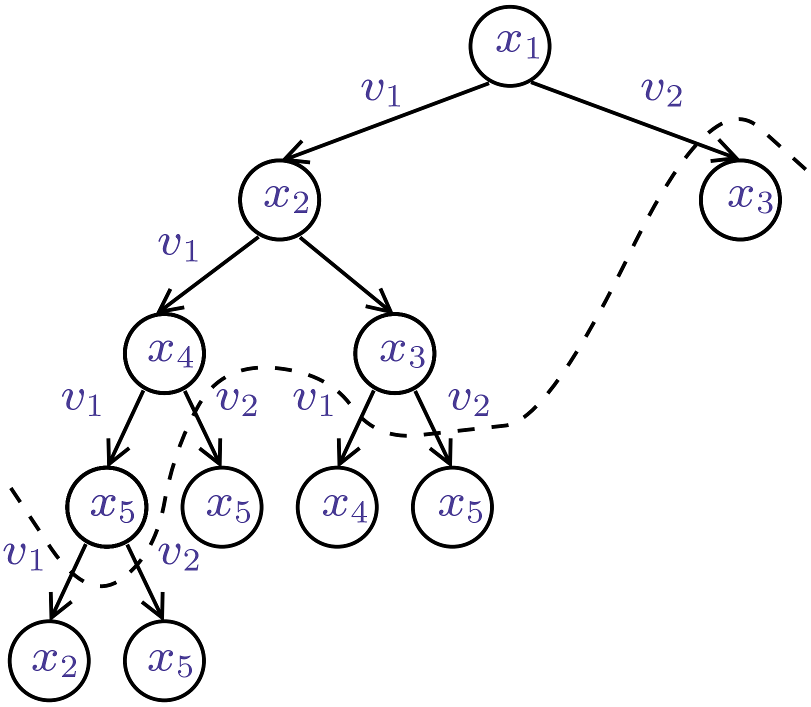
$F^0 := \{x_0\}$
repeat
  $F^{k+1} := F^k \cup \delta(F^k)$
until $F^{k+1} = F^k$
$F_* := F^k$



Complexity: only $O(n \cdot \log n \cdot |V|)$

Variations: Depth-First and Backwards

Depth-first:



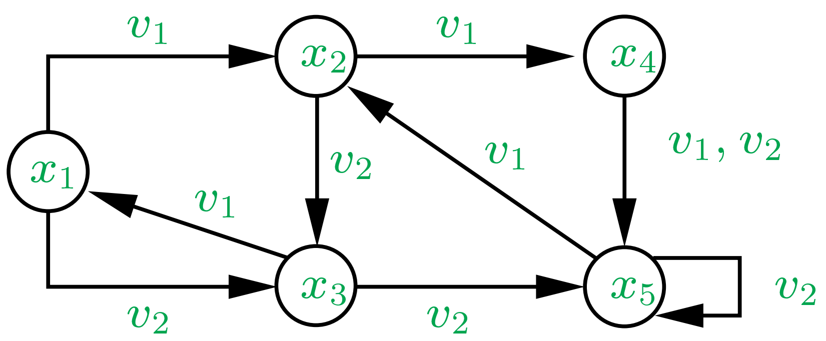
Backwards: find all states from which there is an input leading to P.

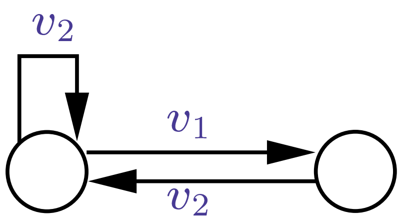
Immediate predecessors:
$\delta^{-1}(x) = \{x' : \exists u \; \delta(x', u) = x\}$

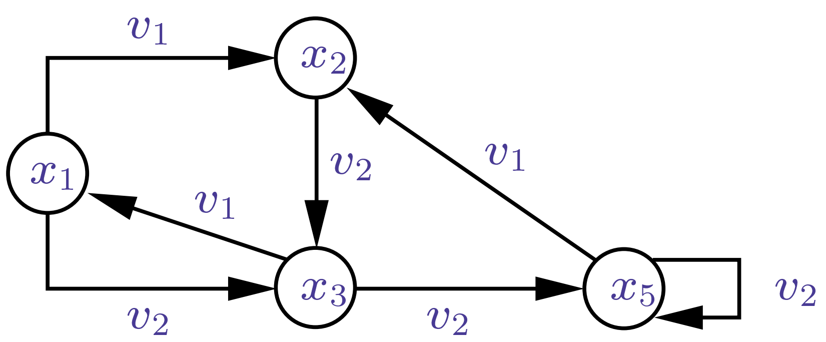
> $F^0 := P$
> **repeat**
> $\quad F^{k+1} := F^k \cup \delta^{-1}(F^k)$
> **until** $F^{k+1} = F^k$
> $F_* := F^k$

Admissible Inputs

So far we have assumed that the external environment can generate all sequences in V^*. Sometimes we have a more restricted environment, e.g. it will never produce $v_1 v_1$. We can build an automaton which models the environment and compose it with the model of the system.







Verification: The State-of-the-Art

There are algorithms that take a description of any open system and verify whether any of the admissible inputs drives the system into a set P. Such algorithms always terminate after a finite number of steps.

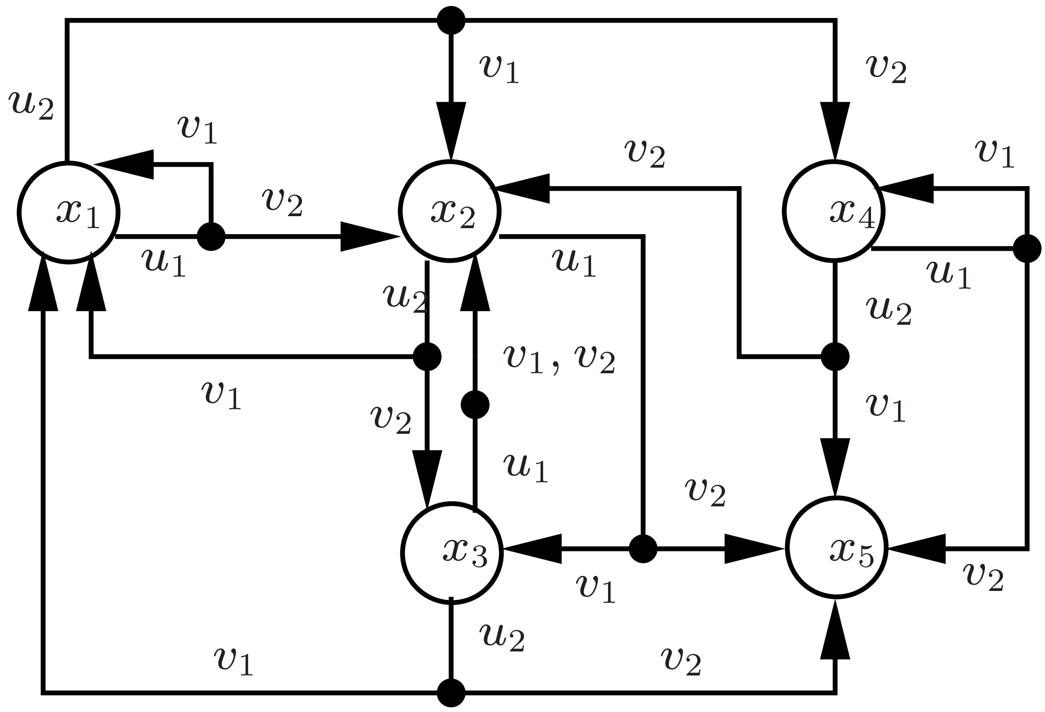
This is essentially what verification is all about.

The result is general: it is valid for every discrete finite-state system. Of course, finite systems can be very large and special tricks are needed to verify them.

The analogue for continuous systems: do the same for a system defined by $\dot{x} = f(x, u)$.

Systems with two Inputs

A two-input transition system is $S = (X, U, V, \delta)$ where X, U and V are finite sets and $\delta : X \times U \times V \to X$ is the transition function.



$$\delta(x_1, u_1, v_1) = x_1 \qquad \delta(x_1, u_1, v_2) = x_2$$
$$\delta(x_1, u_2, v_1) = x_2 \qquad \delta(x_1, u_2, v_2) = x_4$$

The behavior in the presence of two inputs, $\eta \in U^*$ and $\psi \in V^*$: a sequence $\xi(\eta, \psi)$ s.t.

$$\xi[i+1] = \delta(\xi[i], \eta[i], \psi[i])$$

Games and Strategies

Interpretation of inputs:

U: we, the good guys, the controller.

V: they, the bad guys, disturbances.

An antagonist game situation. Our goal is to choose each time an element of U such that the behaviors induces by all possible disturbances are good.

Strategy: a function $c : X^* \to U$

State strategy: a function $c : X \to U$.

Each strategy c converts a type III system into a type II system $S_c = (X, V, \delta_c)$ s.t. $\delta_c(x, v) = \delta(x, c(x), v)$.

Synthesis for Reachability: Let $S = (X, U, V, \delta)$ let $P \subseteq X$ be a set of “bad” states. The controller synthesis problem is: find a strategy c such that all the behaviors of the derived system $S_c = (X, V, \delta_c)$ never reach P.

Finding Winning States and Strategies

Controllable Predecessors: For $S = (X, U, V, \delta)$ and $F \subseteq X$, the set of controllable predecessors of F is

$$\pi(F) = \{x : \exists u \in U \;\; \forall v \in V \;\; \delta(x, u, v) \in F\}$$

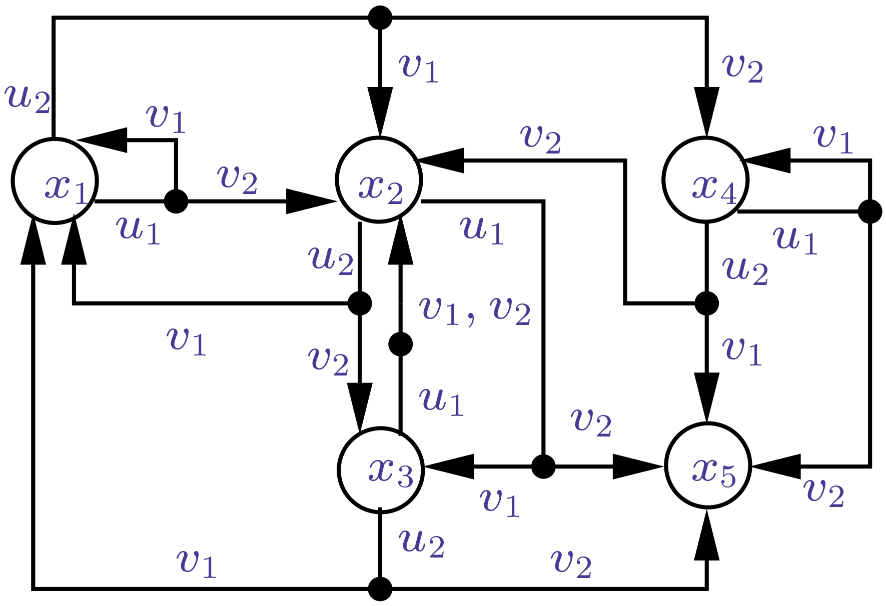
The states from which the controller, by properly selecting u, can force the system into P in the next step.

The following backward algorithm finds the set F_* of "winning states" from which P can be avoided forever.

$F^0 := X - P$
repeat
  $F^{k+1} := F^k \cap \pi(F^k)$
until $F^{k+1} = F^k$
$F_* := F^k$

Remark: this is similar to the Ramadge-Wonham theory of discrete event control.

Synthesis Example

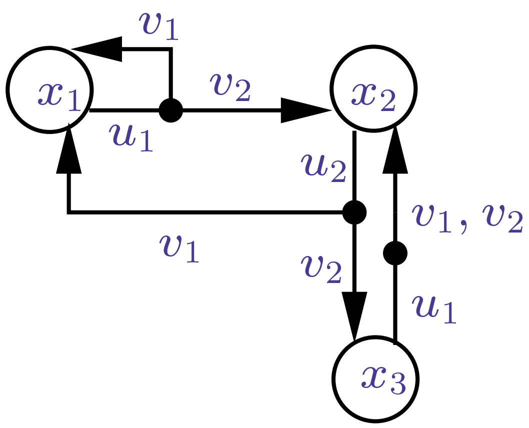


We want to avoid x_5.

$F^0 = \{x_1, x_2, x_3, x_4\}$

$F^1 = \{x_1, x_2, x_3\} = F_*$

The resulting "closed-loop" system always remains in $\{x_1, x_2, x_3\}$.

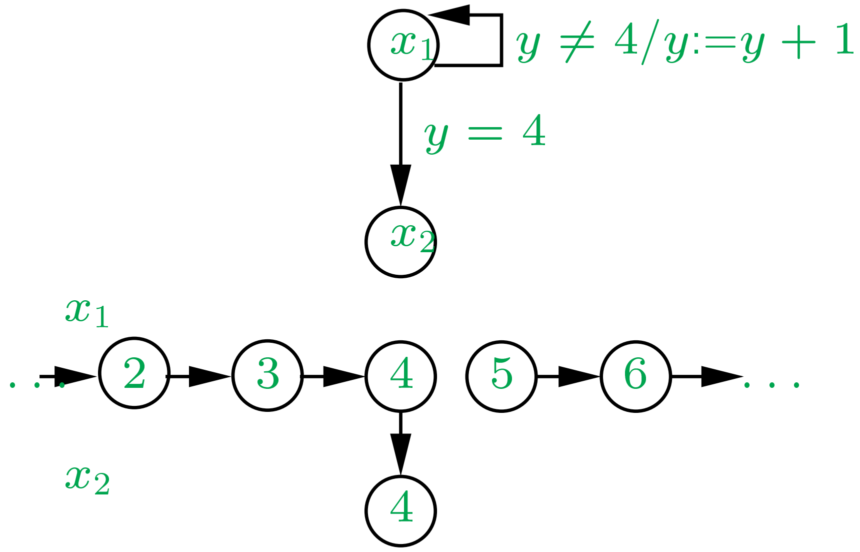


Discrete Infinite-State Systems

Computer program are syntactic representation of dynamical systems with infinite state-space.

repeat
  $y{:=}y+1$
until $y=4$

State space: $\{x_1, x_2\} \times \mathbb{Z}$



Forward reachability algorithm will terminate if started from $(x_1, 2)$ but not from $(x_1, 5)$.

The reachability problem is unsolvable: there is no **general** algorithm that solves every instance of it.

"Deductive" approach: prove properties "analytically".

"Symbolic" approach: reachability using formulae to represent sets of states, e.g. $x = x_1 \wedge y \geq 5$.

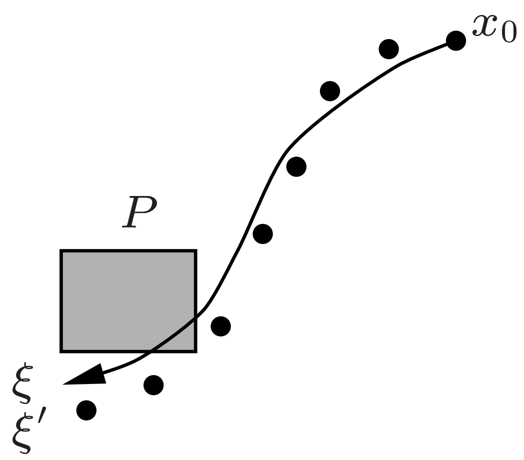
Continuous (and Hybrid) Systems

Why? ...

Problems: state space $\mathbb{R}^n$, infinite even when bounded, time domain $\mathbb{R}$. Mathematical $\mathbb{R}$ vs. numerical $\mathbb{R}$ in the computer.

Reachability for $\dot{x} = f(x)$: When we have a closed-form solution, e.g. for $\dot{x} = Ax$, the reachable set can be written as $F_* = \{x_0 e^{At} : t \geq 0\}$ but how to test whether $F_* \cap P = \emptyset$?

Forward simulation: discretize time and replace the system with $\xi'[(n+1)\Delta] = \xi'[n\Delta] + h(\xi'[n\Delta], \Delta)$.



This is not the "real" thing and it is not guaranteed to converge but that's life.

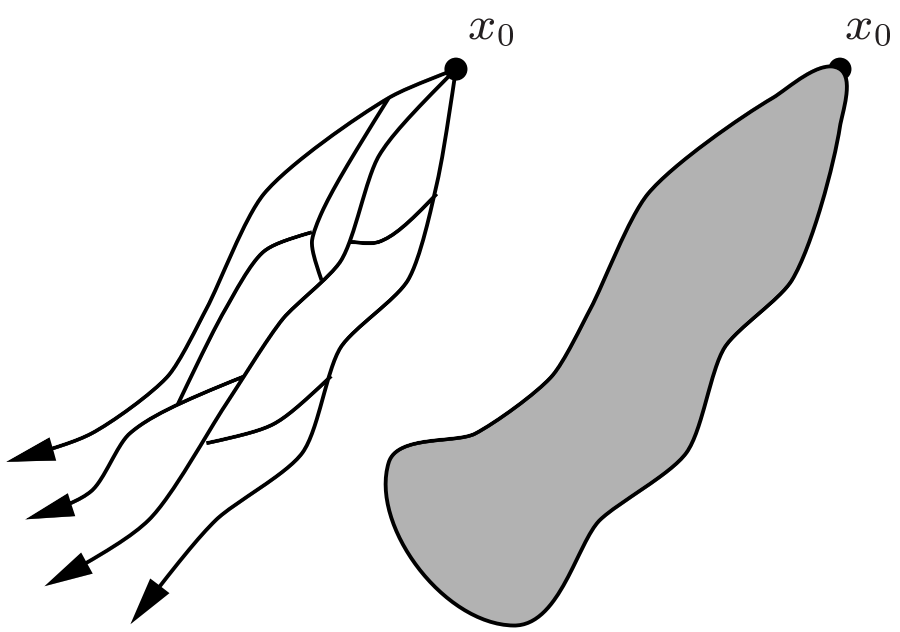
Continuous Systems with Input

Systems of the form $\dot{x} = f(x, v)$. Admissible inputs are signals of the form $\psi : T \rightarrow V$.

Problem: show that no admissible input drives the system into a set P.

For every ψ we can simulate and "compute" $F_*(\psi)$, but there is no finite subset of inputs that covers all reachable states.

The set of all inputs is a **doubly-dense tree**, both vertically (time) and horizontally (V).



Incremental Reachability Computation

Breadth-first computation of reachable states.

$x \xrightarrow{t} x'$ denotes the existence of an input signal $\psi : [0,t] \rightarrow V$ that drives the system from x to x' in t time.

Let F be a subset of X and let I be a time interval. The I-successors of F are all the states that can be reached from F within that time interval, i.e.

$$\delta_I(F) = \{x' : \exists x \in F \ \exists t \in I \ x \xrightarrow{t} x'\}.$$

Semigroup property:

$$\delta_{[0,r_2]}(\delta_{[0,r_1]}(F)) = \delta_{[0,r_1+r_2]}(F).$$

$F^0 := \{x_0\}$
repeat
  $F^{k+1} := F^k \cup \delta_{[0,r]}(F^k)$
until $F^{k+1} = F^k$
$F_* := F^k$

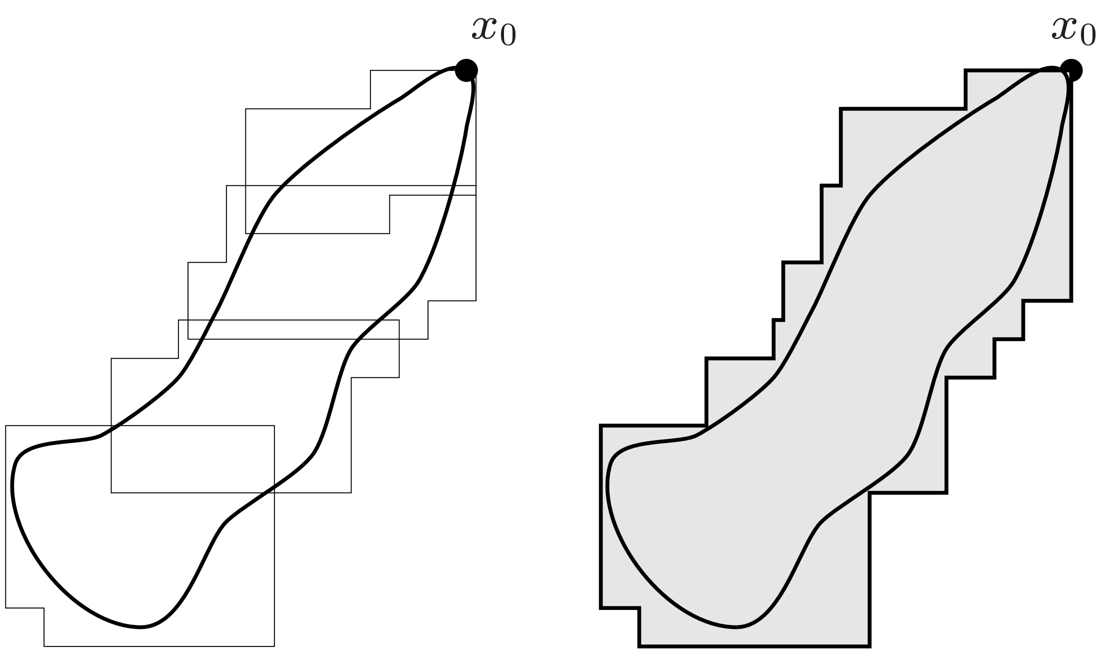
Approximate Reachability Computation

But $\delta_{[0,r]}(F)$ cannot be computed exactly. We can over-approximate it by δ' such that for every F

$$\delta_{[0,r]}(F) \subseteq \delta'_{[0,r]}(F)$$

and $\delta'_{[0,r]}(F)$ belongs to some effective sub-class of $\mathbb{R}^n$, e.g. ppolyhedra.

The result of the algorithm is a set F'_* s.t. $F_* \subseteq F'_*$ and hence $F'_* \cap P = \emptyset$ implies the correctness of the system.



Conclusion

We have developed a system called **d/dt** which accepts as input a description of a continuous or a hybrid system and computes automatically an over-approximation of the reachable states.

More about it in the special session on reachability.

Challenge: use more knowledge on the system dynamics in order to increase the performance and treat systems with higher dimensions.

Challenge: develop algorithms for automatic synthesis of strategies for systems with two inputs, $\dot{x} = f(x, u, v)$.