Control and System Theory of Hybrid Systems

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Outline

- Plan for project.
- Reduction of FDAP systems.
- Triangularization of polytopes.
- Other research topics.

Plan for Project EU.IST.CC

- Control synthesis for hybrid systems based on system theory.
- Decentralized and distributed systems
  (Hybrid systems, DES, Timed DES).
  Modeling, system theory, control, and failure diagnosis.
  (Cooperation with Verimag)
- Engineering control problem with hybrid systems.
  (Cooperation with Parades)

Control and system theory of hybrid systems

Problems

1. For **which subclasses** of hybrid systems can control and system theory be developed which is realistic and nontrivial?
2. **Realization theory** of subclasses of hybrid systems.
3. Which classes of control problems for which subclasses of hybrid systems are decidable and of low complexity,
   and for which of these is the **control law** easy to implement?

Remarks

- Sufficient condition for the existence of
  a finite bisimulation for several classes of hybrid systems.
  (G. Lafferriere, G. Pappas, S. Sastry, MCSS 13 (2000)).
  Based on concept of **O-minimality**.
  Example of a system for which reachability problem is undecidable.
- To be investigated: ...

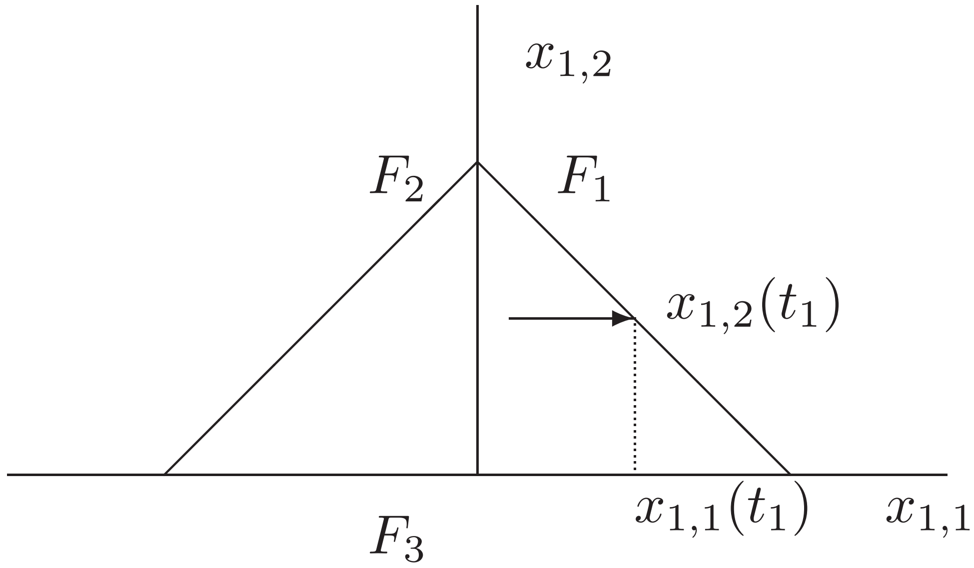
Realization of FDAP systems
Motivation

- Control of piecewise-linear hybrid systems.
- Is affine system on a polytope a minimal realization of its observed trajectories?
- Relation with sufficient conditions for O-minimality.

Example 1. Consider the system,

$$\begin{aligned} \dot{x}_1(t) &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \;\; x_1(t_0) \in X_{1,0} \subseteq X_1. \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_1(t), \;\; X_1 \text{ is triangle.} \end{aligned}$$

Trajectory terminates when it reaches a facet of the polytope and attempts to exit.



Example1 (Continued)

Can the state variable $x_{1,2}$ be eliminated?

Note that:

- State $x_{1,2}$ is neither observable,
- nor controllable.

If $t_1 \in [t_0, \infty)$ is such that the trajectory terminates at that time and either $x(t_1) \in F_1$ or $\in F_2$ then,

$$x_{1,2}(t_1) = \begin{cases} 1 - x_{1,1}(t_1), & \text{if } x_{1,1}(t_1) > 0 \ (\Leftrightarrow \ x_{1,1}(t_1) \in F_1), \\ 1 + x_{1,1}(t_1), & \text{if } x_{1,1}(t_1) < 0 \ (\Leftrightarrow \ x_{1,1}(t_1) \in F_2). \end{cases}$$

Concepts - Polytopes

Def. A **polytope** $P \subset \mathbb{R}^n$ is defined to be
a finite intersection of closed half spaces which, moreover, is bounded,

$$\begin{aligned} P &= \cap_{i=1}^{m} \{x \in \mathbb{R}^n | N_i^T x \leq q_i\}, \\ \dim(P) &= \dim(\mathrm{affh}(P)). \end{aligned}$$

(Equivalently, a polytope is the convex hull of a finite number of vectors.)
Representations,

$$\begin{aligned} P &= \{x \in \mathbb{R}^n | N^T x \leq q\}, \text{ **implicit form;**} \\ &= \{x \in \mathbb{R}^n | \exists y \in S_+^k,\ x = Ay\}, \text{ **explicit form,**} \\ S_+^k &= \{y \in \mathbb{R}^k | \sum_{i=1}^{k} y_i = 1\}. \end{aligned}$$

Faces and facets

Def. Consider polytope

$$P = \{x \in \mathbb{R}^n | N^T x \leq q\}.$$

Face of P defined as the set,

$$\begin{aligned} F &= P \cap \{x \in \mathbb{R}^n | N_s^T x = q_s\}, \\ \dim(F) &= \dim(\mathrm{affh}(F)). \end{aligned}$$

Facet of P is face F such that,

$$\dim(F) = \dim(P) - 1.$$

Notation

$$\begin{aligned} \mathbf{F_{n_P-1}}(P) &= \{F_{n_P-1,i} \subset P | \forall\ i \in Z_r\}, \ \textbf{set of facets,} \\ F_{n_P-1,i} &= P \cap \{x \in \mathbb{R}^n | N_i^T x = q_i\}, \ \textbf{facet}\ i \in \mathbb{Z}_r = \{1, 2, \ldots r\}. \end{aligned}$$

Def. Finite-dim. affine system on a polytope (FDAPS).

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + a, \;\; x(t_0) = x_0 \in X_0 \subseteq X, \\
y(t) &= Cx(t) + Du(t) + c, \\
& T = [t_0, \infty), \quad \textit{time-index set}, \\
& U \subset \mathbb{R}^m, \;\; Y \subset \mathbb{R}^p, \;\; \text{polytopes}, \\
& X \subset \mathbb{R}^n, \;\; \text{closed full-dim. polytope.}
\end{aligned}
$$

Remark Affine character needed because state set $X \subset \mathbb{R}^n$ is also specified.

Problem System reduction of an affine system on a polytope for a set of input-output trajectories

Consider a finite-dimensional affine system on a polytope (System 1)

$$
\begin{aligned}
& X_1 \subset \mathbb{R}^{n_1}, \text{ full-dimensional polytope}, \\
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) + a_1, \ x_1(t_0) = x_{1,0} \in X_{1,0} = X_1, \\
y(t) &= C_1 x_1(t) + D_1 u(t) + c_1.
\end{aligned}
$$

Does there exist another such system (System 2),
affinely reduced with respect to System 1, and
with full-dim. polytope $X_2 \subset \mathbb{R}^{n_2}$, $n_2 < n_1$,
such that their sets of input-output trajectories are equal,

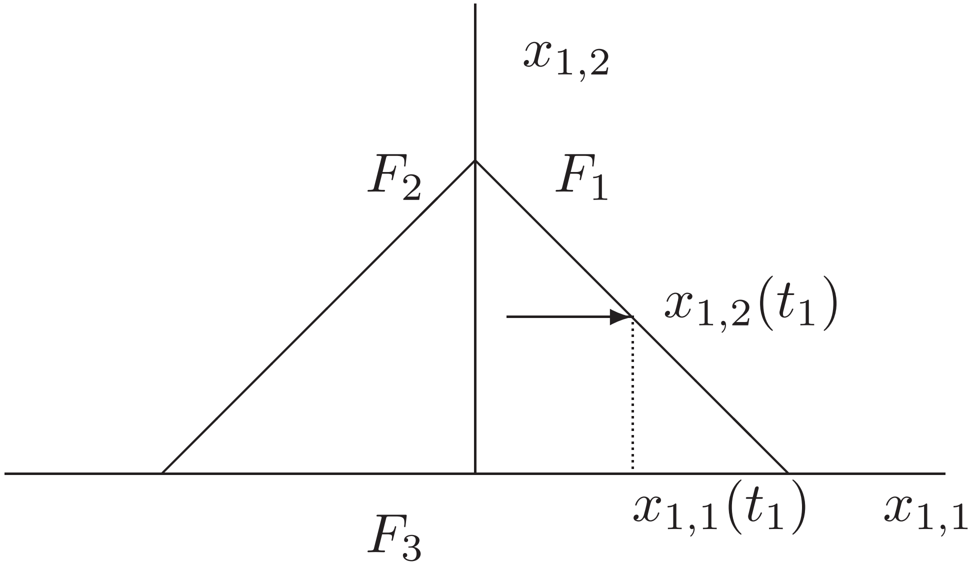
$$UY_1(T) = UY_2(T)?$$

If so, then call System 2 a **realization of System 1's observations which is affinely reduced with respect to** System 1.

Example 1. Consider the system,

$$\begin{aligned}\dot{x}_1(t) &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t),\ x_1(t_0) \in X_{1,0} \subseteq X_1. \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_1(t),\ X_1 \text{ is triangle.}\end{aligned}$$

Can System 1 be affinely reduced?



Remarks

Distinction with realization literature:

- System reduction based on trajectories of a system rather than on impulse response function (also in behavioral approach).
- Time-index set may be finite or infinite depending on the trajectory considered.
- Reduction by affine map rather than linear map.

Def. Consider

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) + a_1,\ x_1(t_0) = x_{1,0} \in X_0 \subseteq X_1 \subseteq \mathbb{R}^{n_1},\\ y(t) &= C_1 x_1(t) + D_1 u(t) + c_1.\end{aligned}$$

Define the **existence interval** of the state, input, and output trajectories,

- $T_1 = [t_0, \infty)$ if $\forall\ t \in [t_0, \infty)$, $x(t) \in X_1$;
- $T_1 = [t_0, t_1]$ if there $\exists\ t_1 \in [t_0, \infty)$ and $\exists\ \epsilon \in (0, \infty)$ such that
  - $F = X_1 \cap \{x \in X_1 | N_i^T x = q_i\}$, called the **exit facet** F of X_1,
  - $x_1(t) \in X_1$ for all $t \in [t_0, t_1)$,
  - $x(t_1) \in F$, and
  - $N_i^T x(s) > q_i$ for $s \in (t_1, t_1 + \epsilon)$.

  State trajectory reaches facet of polytope in finite time and attempts to exit.

Def. (Continued)

Notation Denote of $FDAPS_1$

$$F_{n-1,1}, \ldots, F_{n-1,k}, \text{ the exit facets of } X_1,$$

$$N_1, \ldots, N_k \in \mathbb{R}^{n_1}, \text{ the normals of the exit facets,}$$

$$N_{ExitF} = \begin{pmatrix} N_1^T \\ \vdots \\ N_k^T \end{pmatrix} \in \mathbb{R}^{k \times n_1}.$$

Def.

(a) Define **set of input-output trajectories** of $FDAPS_1$

$$UY(T) = \left\{ \begin{array}{l} (T_1, u, y) \in \text{Pwrset}_{intv.,i}(T) \times (\mathbb{R}^m)^{T_1} \times (\mathbb{R}^p)^{T_1} | \\ \text{either } T_1 = [t_0, \infty) \text{ or } T_1 = [t_0, t_1], \\ u \text{ input, } y \text{ output of FDAPS} \end{array} \right\}.$$

(b) Two FDAP systems $FDAPS_1$, $FDAPS_2$,
realize the same set of input-output trajectories if

$$UY_1(T) = UY_2(T).$$

Def. **Relations on the set of realizations** Consider the $FDAPS_1$,

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) + a_1, \;\; x_1(t_0) = x_{1,0} \in X_{1,0} = X_1, \\ y(t) &= C_1 x_1(t) + D_1 u(t) + c_1. \end{aligned}$$

(a) Define **affine order relation** on the set of realizations of the set of input-output trajectories $UY_1(T)$:

Notation $FDAP_2 \leq FDAP_1$, if

(1) $n_2 \leq n_1$, where $n_i = \dim(X_i)$,

(2) exists affine map $f : X_1 \rightarrow X_2$ which is

(2.1) surjective and

(2.2) for all trajectories,

$(u, y, x_1) \in FDAPS_1$ implies that $(u, y, f(x_1)) \in FDAPS_2$.

Def. (Continued)

(b) Define **affine equivalence relation** on the set of realizations of the set of input-output trajectories:
Notation $FDAPS_1 \equiv FDAPS_2$,
if $n_1 = n_2$ and there exists an affine map, ...

(c) $FDAPS_2$ is said to be **strictly affine reduced with respect to $FDAPS_1$**
if $FDAPS_2 \leq FDAPS_1$ and $FDAPS_2 \not\equiv FDAPS_1$.
Notation $FDAPS_2 < FDAPS_1$.

Theorem

Consider a finite-dimensional affine system on a polytope ($FDAPS_1$),

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) + a_1, \;\; x_1(t_0) = x_{1,0} \in X_{1,0} = X_1, \\ y(t) &= C_1 x_1(t) + D_1 u(t) + c_1. \end{aligned}$$

Note that the initial condition can be arbitrary in the polytope, $X_{1,0} = X_1$.
There exists a realization $FDAPS_2$ of $UY_1(T)$
which is strictly affinely reduced with respect to $FDAPS_1$
($FDAPS_2 < FDAPS_1$) if and only if,

$$X_{1,K} := \ker \begin{pmatrix} C_1 \\ N_{ExitF} \\ \vdots \\ C_1 A_1^{n_1 - 1} \\ N_{ExitF} A_1^{n_1 - 1} \end{pmatrix} \neq \{0\}.$$

Sketch of proof

($\Leftarrow$) $X_{1,K}$ is largest A_1-invariant subspace contained in $\ker(C_1) \cap \ker(N_{ExitF})$, hence commutativity holds in the following diagram,

$$
\begin{tikzcd}
\mathbb{R}^N \arrow[r, "A_1"] \arrow[dd, "\Pi"'] & \mathbb{R}^N \arrow[dr, "C_1"] \arrow[dd, "\Pi"'] & \\
 & & \mathbb{R}^p \\
\mathbb{R}^N / X_{1,K} \arrow[r, "A_2"'] & \mathbb{R}^N / X_{1,K} \arrow[ur, "C_2"'] &
\end{tikzcd}
$$

Sketch of proof (Continued)

$$
\begin{aligned}
X_2 &= \Pi(X_1) \subset \mathbb{R}^{n_1}/X_{1,K}, \text{ is full-dim. polytope,} \\
\dot{x}_2(t) &= A_2 x_2(t) + \Pi B_1 u(t) + \Pi a_1,\ x_2(t_0) = \Pi x_{1,0}, \\
y(t) &= C_2 x_2(t) + D_1 u(t) + c_1,
\end{aligned}
$$

is $FDAPS_2$ which is strictly affinely reduced with respect to $FDAPS_1$ ($n_2 < n_1$).

State trajectories simultaneously on exit facets of respective polytopes.

($\Rightarrow$) Affine map $f : X_1 \to X_2$, $f(x) = Sx + q$, $\mathrm{rank}(S) = n_2$.

$X_{1,K}$ is the largest A_1-invariant subspace contained in $\ker(C_1) \cap \ker(N)$,

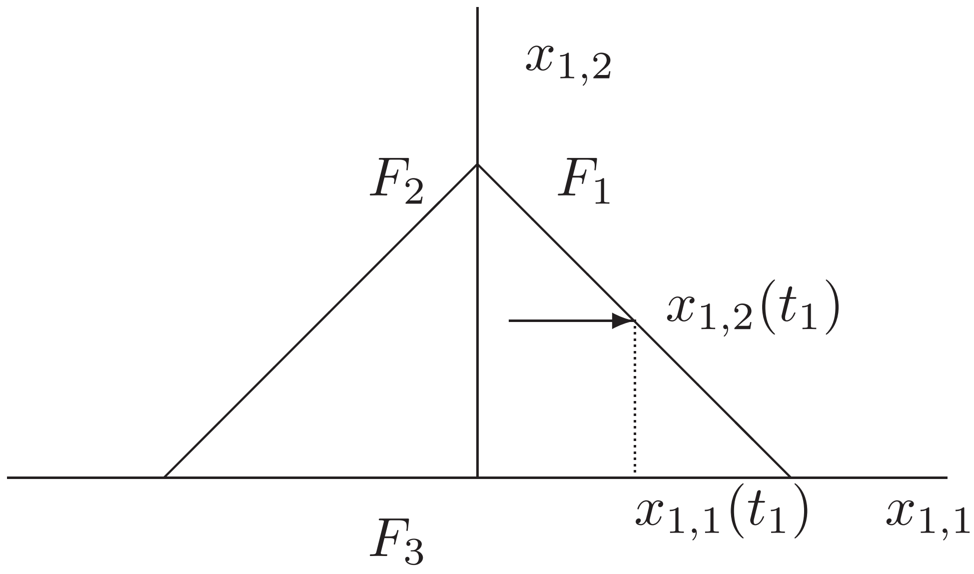
$\{0\} \neq \ker(S)$ is an A_1-invariant subspace contained in $\ker(C_1) \cap \ker(N)$,

hence $\{0\} \neq \ker(S) \subseteq X_{1,K}$, $X_{1,K} \neq \{0\}$.

Example 1 No reduction possible (Continued)

Note that the exit facets are F_1, F_2, and,

$$
\begin{aligned}
N_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T, \; N_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}^T, \\
X_{1,K} &\subseteq \ker(N_1^T) \cap \ker(N_2^T) = \{0\}.
\end{aligned}
$$



Example 2 Reduction possible

Consider $FDAPS_1$,

$$
\begin{aligned}
X_1 &= [-1,+1]^2 \subset \mathbb{R}^2, \\
\dot{x}_1(t) &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t),\ x_1(t_0) = x_{1,0} \in X_{1,0}, \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_1(t).
\end{aligned}
$$

This system can be strictly affinely reduced ($FDAPS_2 < FDAPS_1$) because,

$$
X_{1,K} =< \begin{pmatrix} 0 \\ 1 \end{pmatrix} > \neq \{0\}.
$$

Thus the state variable $x_{1,2}$ can be eliminated.

Further research on realization of FDAP systems

1. Reduction of FDAP systems due to lack of reachability/controllability. (Nondecidability of reachability for subclass, relation with O-minimality)
2. Realization of FDAP systems.
3. Use of these results for control of piecewise-linear hybrid systems.

Triangularization of a polytope

Motivation

Needed for Problem Control-to-Facet. Literature:

- C.W. Lee, *Subdivisions and triangulations of polytopes*, 1997.
- K. Fukuda, *Frequently asked questions in polyhedral computation*, 2000.
- B. Büeler and A. Enge and K. Fukuda, *Exact volume computation for convex polytopes: A practical study*.

Software delaunayn in MATLAB.
Time complexity is $O(m^{\lceil n/2 \rceil})$, of polytope in $\mathbb{R}^n$ with m vertices.
Upper bound on simplices $O(m^{\lfloor (n+1)/2 \rfloor})$.

Triangulation

MATLAB Program delaunayn.

n, $\mathbb{R}^n$	m vertices	Time	Simplices
10	20	seconds	1000 - 2000
10	30	20 s.	20.000

Fukuda (1998) states that triangulation is tractable for polytopes in $n = 15$ with (hyperplanes, vertices) $= (10^2, 10^4)$ or $(10^4, 10^2)$.

Example of triangularization of a polytope

Triangulation of the unit cube in $\mathbb{R}^4$ from the literature.

Yields a triangulation into 16 simplices.

$$
\begin{aligned}
T_1 &= [0,1,2,4,8] & \qquad T_9 &= [1,2,4,8,14] \\
T_2 &= [4,8,12,13,14] & T_{10} &= [1,4,8,13,14] \\
T_3 &= [2,8,10,11,14] & T_{11} &= [1,2,8,11,14] \\
T_4 &= [2,4,6,7,14] & T_{12} &= [1,2,4,7,14] \\
T_5 &= [1,8,9,11,13] & T_{13} &= [1,8,11,13,14] \\
T_6 &= [1,4,5,7,13] & T_{14} &= [1,4,7,13,14] \\
T_7 &= [1,2,3,7,11] & T_{15} &= [1,2,7,11,14] \\
T_8 &= [7,11,13,14,15] & T_{16} &= [1,7,11,13,14]
\end{aligned}
$$

Other topics of research

- Asynchronous timed discrete-event system and decentralized failure diagnosis.
- Decentralized failure diagnosis with communication between diagnosers.
- Quantization of a control law for the control-to-facet problem.