Efficient Mode Enumeration of Compositional Hybrid Models

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Modeling Requirements

Linear dynamics changes according to

- Logic state
- Exogenous logic inputs
- Threshold conditions
- Time
- Any logic combination of the former

Example:

\[
\begin{align*}
\text{IF } & \left( (t > T_1) \lor (x_1 \leq 7) \right) \land (u_1 = \text{TRUE}) \\
\text{THEN } & x(t + 1) = A_j x(t) + B_j u(t) + f_j
\end{align*}
\]
Discrete Hybrid Automata
Switched Affine Systems

Linear affine dynamics depends upon the mode selector $i(t)$

\[ x'_r(k) = A_i(k)x_r(k) + B_i(k)u_r(k) + f_i(k) \]

\[ y_r(k) = C_i(k)x_r(k) + D_i(k)u_r(k) + g_i(k) \]
Event Generator

Generates a logic signal according to the satisfaction of a linear affine constraint

\[ \delta_e(k) = f_H(x_r(k), u_r(k), k) \]
Finite State Machine

Discrete dynamic process
Evolves according to a logic state update function

\[ x_b'(k) = f_B(x_b(k), u_b(k), \delta_e(k)) \]
\[ y_b(k) = g_B(x_b(k), u_b(k), \delta_e(k)) \]
A Boolean function selects the active mode \( i(k) \) of the SAS

\[
i(k) = f_M(x_b(k), u_b(k), \delta_e(k))
\]
DHA and Other Modeling Frameworks

Piecewise Affine Models (PWA) define an affine dynamics on each cell of a polyhedral partition.

Mixed Logic Dynamical Models (MLD) are linear systems plus mixed integer inequalities.

\[
\begin{align*}
x'(k) & = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\
y(k) & = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\
E_2\delta(k) + E_3z(k) & \leq E_1u(k) + E_4x(k) + E_5
\end{align*}
\]
Composition of DHAs

More DHAs can be combined to form a compositional DHA

Many times hierarchical structures cannot be identified

But interaction among the parts removes modes and reduces complexity
Problem Statement

Enumerate all the possible modes of a given composition of DHAs,

Number of possible modes could be exponential in the number of composing systems
Let $\mathcal{A} = \{H_i\}_{i=1}^{n}$, $H_i = \{x : a_i x - b_i = 0\}$ be a collection of $n$ hyperplanes in $\mathbb{R}^d$.

**Theorem** Each polyhedral region (or cell) is associated to a sign marking.

**Theorem** The total number of cells is bounded by Buck’s formula

$$\# M \leq \sum_{i=0}^{d} \binom{n}{i}$$
Hyperplane Arrangements - Algorithms

There is an optimal algorithm for enumeration of hyperplane arrangements with time and space complexity $O(n^d)$ (Edelsbrunner ‘87).

There is reverse search algorithm (Fukuda ‘96, ‘01) for enumeration of hyperplane arrangements that runs in $O(n \text{ lp}(n,d) \#M)$ time and $O(n,d)$ space, where $\text{lp}(n,d)$ is the complexity of solving a linear program with $d$ variables and $n$ constraints.

Let $M=\text{hyparr}(\mathcal{A}, \mathcal{R})$ be a function that computes all the sign markings of the hyperplane arrangement $\mathcal{A}$ in region $\mathcal{R}$. Let $\mathcal{P}_m, m \in M$ be the polyhedron associated with the sign marking $m$. 
Single DHA

The Event Generator defines a hyperplane arrangement on the input+state space. For any discrete state, discrete input and cell of the arrangement there is exactly one feasible mode.
Connection of DHAs is an oriented graph

Replace the loops with constraints \((u=y)\)

Determine the computational order \(\mathcal{O}\) by topological sorting
function enum(\mathcal{R},i),
if \exists \Sigma_{O(i)}, \mathcal{R} \neq \emptyset
M=\text{hyparr}(\text{EG}_{\mathcal{O}(i)}, \mathcal{R});
\text{foreach } m \in M, 
Q = Q \cup \text{enum}(\mathcal{P}_{m,i});
else
\text{if sat}(\mathcal{R})
\text{return } \{ \mathcal{R} \}
else
\text{return } \emptyset
\end{align*}

Q = \emptyset
\text{foreach } x_{b,i}, u_{b,i}
Q = Q \cup \text{enum}(\mathbb{R}^d, 1);
Composition of DHAs - Example

Let $\Sigma_3$ have $u \in \mathbb{R}$, $x \in \mathbb{R}$, $\Sigma_1$ be a static map
Application: Efficient PWA conversion

For any discrete state, discrete input and cell of the arrangement there is exactly one feasible mode, ⇒ PWA model

Similar to Bemporad ’02
Example: Car

Renault Clio 1.9 DTI RXE

Continuous (gas pedal, and brakes) and discrete (gear ratio) inputs

30 regions and 6 modes enumerated in 7.5 s on PC 650MHz
Model Predictive Control (MPC) amounts to apply optimal control in receding horizon.
Example: Paperboy

Deliver newspapers to 2 households
Piecewise affine slope hill
Uses MPC(!!) to optimize his trajectory

Model: 3+8+0 constraints in the EGs, 4·42 cells, 36 modes
MLD model: 132 constraints, 21 integer variables
Example: Paperboy

Homemade MIQP solvers allow full control of branching strategies ⇒ speedup factor 210 with prediction horizon 3

Commercial solvers (i.e. CPLEX) have less freedom, and usually include a rich bag of branching heuristics ⇒ add cuts

Advantage appears evident with long prediction horizons
Conclusions

DHA models:
- Capture hybrid phenomena
- Linear dynamics
- Logic-, threshold-, and time-based switching

Cell/Region/Modes enumeration:
- A tool from computational geometry may help in the hybrid domain
- Compute an equivalent PWA model
- Reduces the complexity of MPC

Open Problems:
- Exploit sign information to merge cell with the same dynamics
- Extension to other applications (continuous time systems)